a = [a,H] = aH-Ha—oh, my, gotcha!!!

The ladder operator solution to the simple harmonic oscillator problem is subtle, exquisite, and rather slippery—so I thought you might appreciate a recapitulation of what I said in class . . . . You might want to go through the argument line-by-line until it clicks!

There were three steps in the argument:

1. The first step was to show that the eigenvalues of the Hamiltonian H are equal to $\hbar\omega$ times $\frac{1}{2}$ plus the eigenvalues of the number operator $N = a^\dagger a$ (which will turn out to be n, so we will end up with $(n + \frac{1}{2})\hbar\omega$). We did this by showing that the Hamiltonian H is $\hbar\omega$ times the sum of the number operator plus one half the identity operator,

$$H \;=\; \big(a^\dagger a + \frac{1}{2}\big)\;\hbar\omega.$$

We showed this by defining the $a^\dagger$ and a operators, and then calculating $a^\dagger a$. Note that once we found that $H = (a^\dagger a + \frac{1}{2})\;\hbar\omega$, we immediately knew that the eigenvectors of H would be the same as the eigenvalues of $a^\dagger a$—because every vector is an eigenvector of the identity operator! We also immediately knew that the eigenvalues of H would be equal to $\hbar\omega$ times the eigenvalues of the $a^\dagger a$ operator plus $\frac{1}{2}\hbar\omega$.

2. The second step was to show that when the $a^\dagger$ and a operators act on any eigenvector of H, we get back another eigenvector of H one step up or down the ladder of states. We showed this by calculating the three commutators:

$$\begin{aligned}[][a, a^\dagger] &= +1\\ [a, H] &= +a\\ [a^\dagger, H] &= -a^\dagger\end{aligned}$$

and considering the action of the last two commutators on any eigenvector of the Hamiltonian

$$\begin{aligned}[][a, H]\;|\text{eigenvector of H}\!> &= +a\;\;|\text{eigenvector of H}\!>\\ [a^\dagger, H]\;|\text{eigenvector of H}\!> &= -a^\dagger\;|\text{eigenvector of H}\!>.\end{aligned}$$

By expanding the commutators, we found

$$\begin{aligned}\big(aH - Ha\big)\;|\text{eigenvector of H}\!> &= +a\;\;|\text{eigenvector of H}\!>\\ \big(a^\dagger H - Ha^\dagger\big)\;|\text{eigenvector of H}\!> &= -a^\dagger\;|\text{eigenvector of H}\!>.\end{aligned}$$

which allowed us to conclude that

$$\begin{aligned}a\;|\epsilon\!> &= (\epsilon - 1)\;|\epsilon - 1\!>\\ a^\dagger\;|\epsilon\!> &= (\epsilon + 1)\;|\epsilon + 1\!>.\end{aligned}$$

This showed us that the eigenvalues of H are separated by $\pm\hbar\omega$. Combining this with the $\frac{1}{2}\hbar\omega$ from step one, we then concluded that the eigenvalues of the Hamiltonian are given by

$$E_n \;=\; \left(n + \frac{1}{2}\right)\hbar\omega$$

where n is any integer (*i.e.,* positive, negative, or zero!!!). However, in step three, we found the smallest eigenvalue of the number operator is equal to zero.

So the only other thing we did not know yet was whether the raising and lowering operators return normalized eigenvectors of the Hamiltonian, *i.e.,* are the vectors $a|\epsilon>$ and $a^\dagger|\epsilon>$ normalized eigenvectors of H? We did know that they are eigenvectors of H with eigenvalues of $(\epsilon-1)\hbar\omega$ and $(\epsilon+1)\hbar\omega$, respectively, but we did not know whether they are normalized—and, in fact, they are not!

3. The third step was to calculate the normalization coefficients. To do this we started with two adjacent normalized states, $|n> \equiv |E = (n+\frac{1}{2})\hbar\omega>$ and $|n-1> \equiv |E = ((n-1)+\frac{1}{2})\hbar\omega>$ and then we calculated the expectation value of the number operator in two different ways:

(i) First, we started with the lowering operator equation

$$a\;|n> = \;c_n\;|n-1>$$

and then we calculated the adjoint of this equation

$$<n|\;a^\dagger \;= <n-1|\;c_n^*.$$

We combined these to evaluate the expectation value of the number operator

$$\begin{aligned} <n|\;a^\dagger a\;|n> \;&= <n-1|\;c_n^* c_n\;|n-1> = \;|c_n|^2\;\;<n-1|n-1> \\ &= \;|c_n|^2. \end{aligned}$$

(ii) Second, we replaced $a^\dagger a$ by $\widehat{H} - \frac{1}{2}$ and recalculated the expectation value of N

$$\begin{aligned} <n|\;\widehat{H} - \frac{1}{2}\;|n> \;&= <n-1|\;\left[\;(n+\frac{1}{2}) + \frac{1}{2}\;\right]\;|n> = \;n\;\;<n|n> \\ &= \;n. \end{aligned}$$

By combining these two calculations, we found

$$\begin{aligned} |c_n|^2 = n \;\Rightarrow\; c_n = \sqrt{n} \\ \Rightarrow\;\; a\;|n> = \sqrt{n}\;|n-1>\,. \end{aligned}$$

Finally, to see that the lowest eigenvalue of the number operator is zero, we considered

$$a\;|0> = \;\sqrt{0}\;|0-1> = \;0\;|-1> = \;\vec{0}.$$

So $|0>$ is the bottom rung on the ladder (lowering it we obtain the zero vector), and consequently the lowest eigenvalue of H is $\frac{1}{2}\hbar\omega$, which is the zero point energy of the oscillator.

LECTURE 19 7-24-07

SOLVED TISE IN POSITION SPACE

$$H|n\rangle = E_n |n\rangle$$

WAVE FUNCTIONS $\varphi_n(x)$ IN X-SPACE

OPERATOR IN X-SPACE

EIGENENERGIES

FOUND $E_n = (n + 1/2)\hbar\omega$ $n = 0, 1, 2, 3, \ldots$

$$\varphi_n(y) = H_n(y)\, e^{-y^2/2}$$

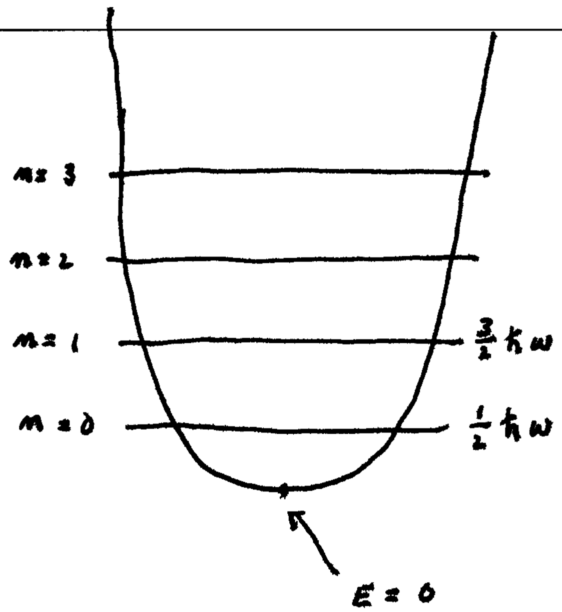
STATIONARY STATES

AKA

ENERGY EIGENFUNCTIONS

HERMITE POLYNOMIALS

GAUSSIAN



LADDER OF EQUALLY SPACED STATES

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

SPACING = $\hbar\omega$

GROUND STATE

$$E_0 = \tfrac{1}{2}\hbar\omega$$

ZERO POINT ENERGY

ZERO POINT MOTION

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

$\Rightarrow$ SYSTEM CAN NEVER STOP

FIND TWO LINEAR OPERATORS

a^+ = RAISING OPERATOR = CREATION OPERATOR

a = LOWERING OPERATOR = DESTRUCTION OP

IN FIELD THEORY

PHONONS

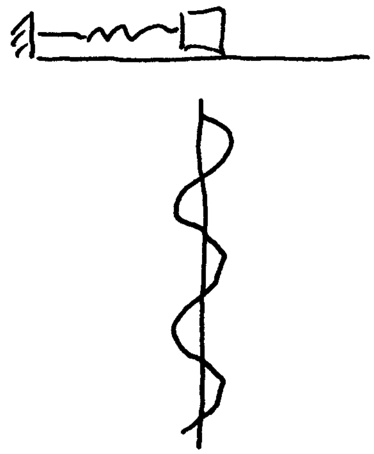
PHOTONS

ELECTRONS

⋮

FEYNMAN STORY

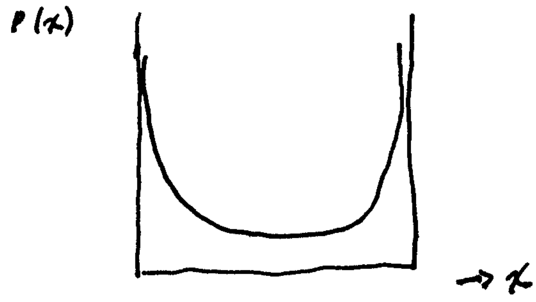
CLASSICAL HARMONIC OSCILLATOR



$P(x)$ "STOPS" AT TURNING POINTS $\Rightarrow$ $P(x)$ LARGEST

FASTEST AT CENTER $\Rightarrow$ $P(x)$ SMALLEST

$$P(x) \sim \frac{dt}{dx} = \frac{1}{v}$$



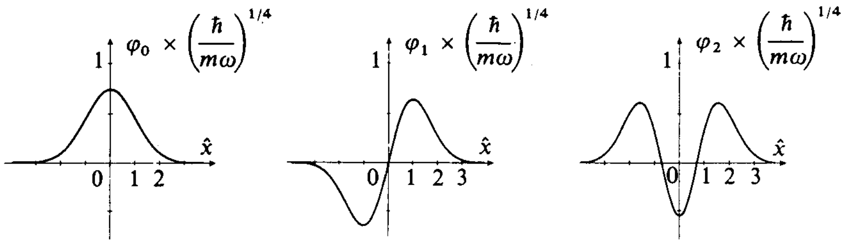


FIGURE 4

Wave functions associated with the first three levels of a harmonic oscillator.

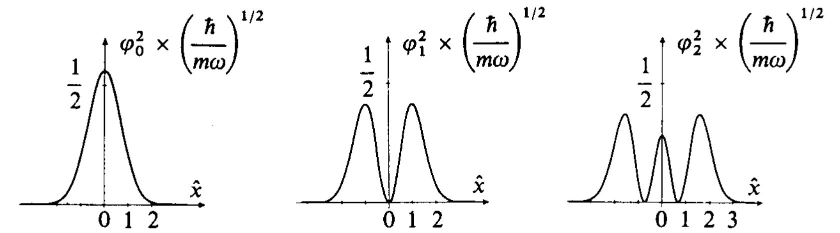


FIGURE 5

Probability densities associated with the first three levels of a harmonic oscillator.

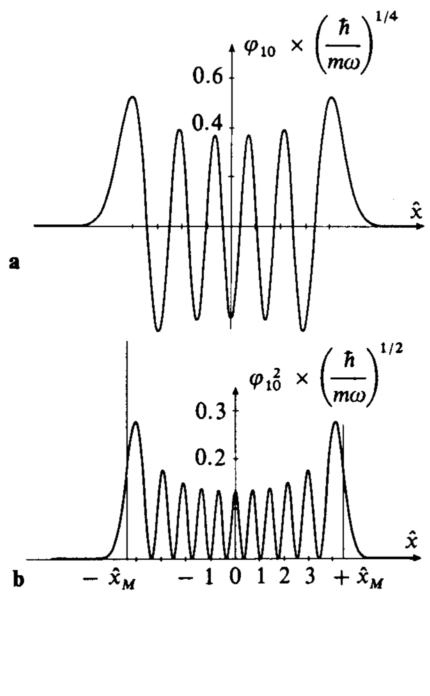


FIGURE 6

Shape of the wave function (fig. a) and of the probability density (fig. b) for the $n = 10$ level of a harmonic oscillator.

$$
\begin{aligned}
H_0(\xi) &= 1 \\
H_1(\xi) &= 2\xi \\
H_2(\xi) &= 4\xi^2 - 2 \\
H_3(\xi) &= 8\xi^3 - 12\xi \\
H_4(\xi) &= 16\xi^4 - 48\xi^2 + 12 \\
H_5(\xi) &= 32\xi^5 - 160\xi^3 + 120\xi \\
H_6(\xi) &= 64\xi^6 - 480\xi^4 + 720\xi^2 - 120 \\
H_7(\xi) &= 128\xi^7 - 1344\xi^5 + 3360\xi^3 - 1680\xi \\
H_8(\xi) &= 256\xi^8 - 3584\xi^6 + 13440\xi^4 - 13440\xi^2 + 1680 \\
H_9(\xi) &= 512\xi^9 - 9216\xi^7 + 48384\xi^5 - 80640\xi^3 + 30240\xi \\
H_{10}(\xi) &= 1024\xi^{10} - 23040\xi^8 + 161280\xi^6 - 403200\xi^4 + 302400\xi^2 \\
&\qquad - 30240.
\end{aligned}
\tag{11–23}
$$

ENERGY EIGEN FCNS: COMPLETE SET

ORTHOGONAL SET

NORMAL SET

$$\psi_n(q) = A_n H_n(q)\, e^{-q^2/2}$$

$$\int \psi_n(q)\, \psi_m(q)\, dq = 1 \quad \text{if } n=m$$

$$= 0 \quad \text{otherwise}$$

$$= \delta_{nm} \quad \text{KRONECKER DELTA}$$

$$\text{ANY FCN}(q) = \sum a_n \psi_n(q)$$

$$a_n = \int_{-\infty}^{\infty} \text{ANY FCN}^*(q)\, \psi_n(q)\, dq$$

NEED THREE COMUTATORS:

$$[a, a^+] = 1$$

$$[a, H] = a \quad \Rightarrow \quad aH - Ha = a$$

$$[a^+, H] = -a^+ \quad \Rightarrow \quad a^+H - Ha^+ = -a^+$$

$$aH - Ha = a \quad \Rightarrow \quad Ha = aH - a$$

$$Ha|\epsilon\rangle = (aH - a)|\epsilon\rangle$$

$$H|\epsilon\rangle = \epsilon|\epsilon\rangle$$

$$= (a\epsilon - a)|\epsilon\rangle$$

$$H|\epsilon-1\rangle = (\epsilon-1)|\epsilon-1\rangle$$

$$= (\epsilon-1)\left[a|\epsilon\rangle\right]$$

$$H|\epsilon+1\rangle = (\epsilon+1)|\epsilon+1\rangle$$

SUMMARY

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$H = (a^+a + 1/2)\,\hbar\omega$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle$$

$\longrightarrow$ generating fcn

NUMBER OPERATOR

$$a^+a = N$$

$$N = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$

LADDER OPERATORS

$$a\,|m\rangle = \sqrt{m}\,|m-1\rangle \qquad \text{DOWN}$$

$$a^{+}|m\rangle = \sqrt{m+1}\,|m+1\rangle \qquad \text{UP}$$

TRANSLATE ALL THIS INTO MATRIX LANGUAGE

$$a\,|m\rangle = \sqrt{m}\,|m-1\rangle$$

$$a = \begin{bmatrix} 0 & \sqrt{1} & & & \\ 0 & 0 & \sqrt{2} & & 0 \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ & & & & \end{bmatrix} \begin{array}{l} \left[\begin{array}{c} \\ \\ \\ \\ \end{array}\right] \end{array} \begin{array}{l} m=0 \\ m=1 \\ m=2 \\ \\ \end{array}$$

MATRIX ELEMENTS

$$\langle m'|a|m\rangle = \langle m'|\sqrt{m}\,|m-1\rangle$$

$$= \sqrt{m}\;\delta_{m',\,m-1}$$

$$a^+ = \begin{bmatrix} 0 & & & & \\ \sqrt{1} & 0 & & \text{O} & \\ 0 & \sqrt{2} & 0 & & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$\langle n' | a^+ | n \rangle = \sqrt{n+1}\ \delta_{n', n+1}$$

$$x_{op} = \sqrt{\frac{\hbar}{2m\omega}}\ (a + a^+)$$

$$x_{op} = \begin{bmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ & & & & \ddots \end{bmatrix} \sqrt{\frac{\hbar}{2m\omega}}$$

$$P_{op} = i\sqrt{\frac{m\hbar\omega}{2}}\left[a^{+} - a\right]$$

$$P_{op} = i\sqrt{\frac{m\hbar\omega}{2}}\begin{bmatrix} 0 & -\sqrt{1} & & & \\ \sqrt{1} & 0 & -\sqrt{2} & & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \\ & & & & \\ & & & & \end{bmatrix}$$

$$H = \begin{bmatrix} 1/2 & & & & \\ & 3/2 & & & \\ & & 5/2 & & \\ & & & 7/2 & \\ & & & & \ddots \end{bmatrix}\hbar\omega$$

TO FIND $\psi_0(x)$

$$a|0\rangle = 0$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x_{op} + i\sqrt{\frac{1}{2m\omega\hbar}}\, p_{op}$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\, x + i\sqrt{\frac{1}{2m\omega\hbar}}\left(-i\hbar\frac{\partial}{\partial x}\right)$$

CHANGE VARIABLES

$$y = \sqrt{\frac{m\omega}{2\hbar}}\, x$$

$$dy = \sqrt{\frac{m\omega}{2\hbar}}\, dx$$

$$a \rightarrow \frac{1}{\sqrt{2}}\left[y + \frac{d}{dy}\right]$$

$$a|0\rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}\left[y + \frac{d}{dy}\right]\psi_0(y) = 0$$

FIRST ORDER DIFF EQ!

$$\frac{d\psi_0(y)}{\psi_0(y)} = -y\,dy$$

$$\ln \psi_0 = -\frac{y^2}{2} + c$$

$$\psi_0(y) = e^{c}\, e^{-y^2/2} = A_0\, e^{-y^2/2}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

TO CALCULATE HIGHER STATES

$$a^+ = \frac{1}{\sqrt{2}}\left[y - \frac{d}{dy}\right]$$

$$\psi_n(y) = \frac{(a^+)^n}{\sqrt{n!}}\,\psi_0(y)$$

$$\psi_n(y) = H_n(y)\, e^{-y^2/2}$$

Gaussian

Hermite polynomials