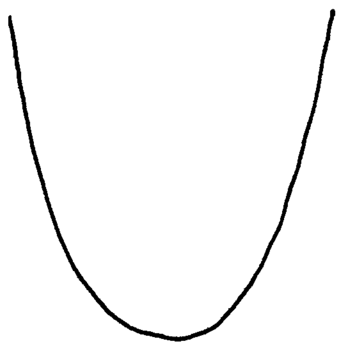
LECTURE 22

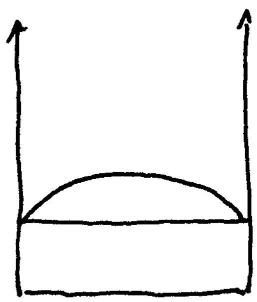
7-30-09

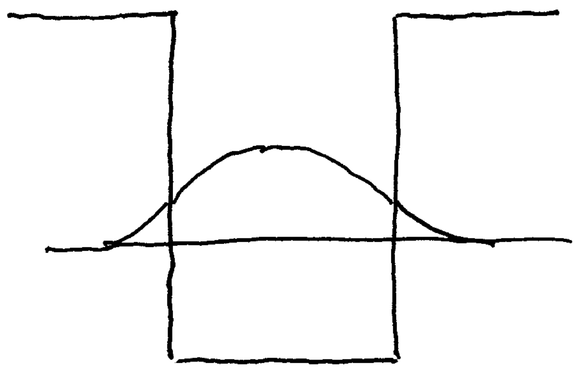


$V(x) = \frac{1}{2} k x^2$

FIRST NON-TRIVIAL $V(x)$

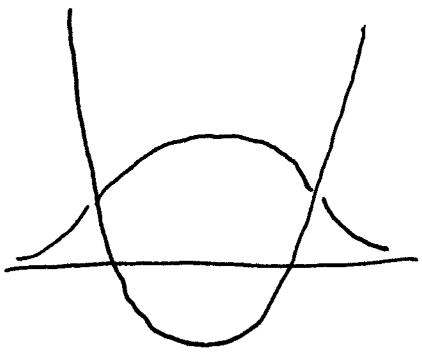
WHAT SHOULD WE EXPECT?





$E > V$ sin inside

$E < V$ exp tails outside

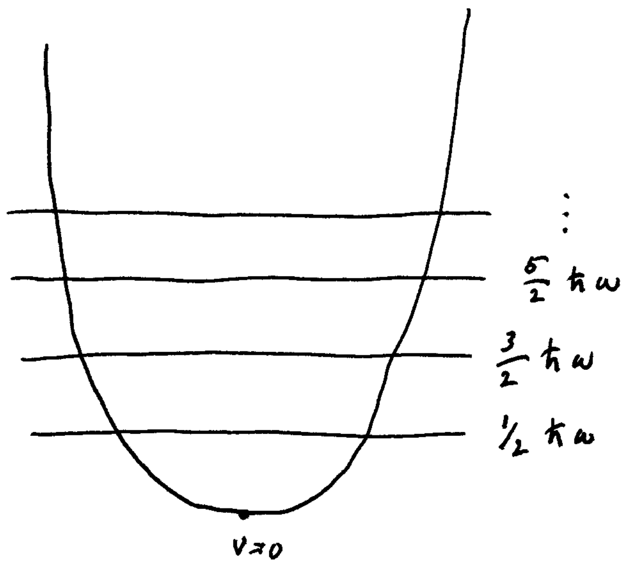


inside $E > V$ but variable

outside $E < V$ but variable

ground state => no nodes

WHAT WE WILL FIND



EQUALLY SPACED ENERGY LEVELS

$\Delta E = \hbar\omega$

GROUND STATE $E = \frac{1}{2}\hbar\omega$

STATIONARY STATES

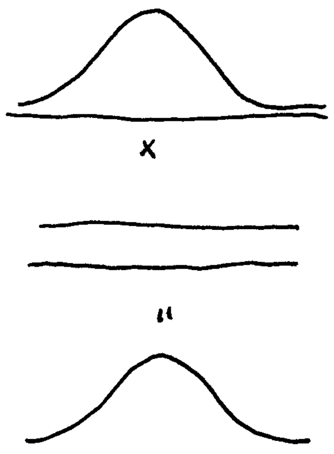
$$\psi_n(x) = (n\text{-TH ORDER POLYNOMIAL})(\text{GAUSSIAN})$$

HERMITE POLYNOMIALS — ASYMPTOTIC DECAY

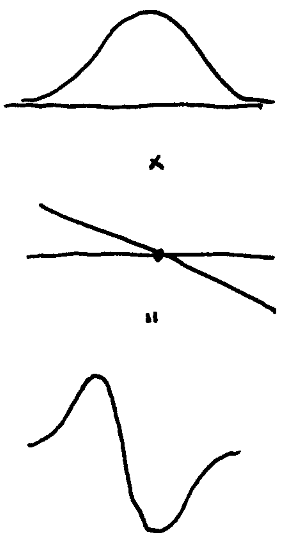
$$= A_n H_n(x)\, e^{-cx^2/2}$$

$$\psi_n(x) = A_n H_n(y)\, e^{-y^2/2}$$

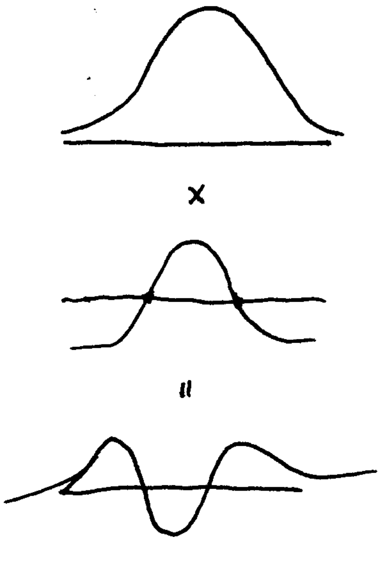
CARTOON VERSION



$\psi_0(x)$



$\psi_1(x)$



$\psi_3(x)$

UBIQUITOUS

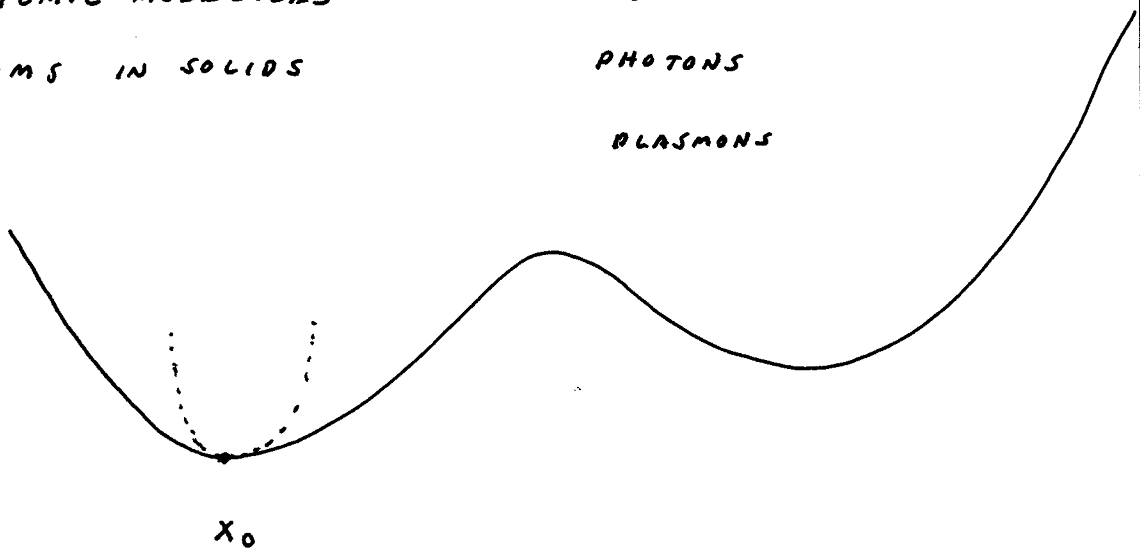
DIATOMIC MOLECULES

ATOMS IN SOLIDS

PHONONS

PHOTONS

PLASMONS



$$V(x) = V_0 + \left.\frac{\partial V}{\partial x}\right|_{x_0} (x - x_0) + \frac{1}{2!} \frac{\partial^2 V}{\partial x^2} (x - x_0)^2$$

$V_0 = 0$

$x_0 = 0$

$$V(x) = \frac{1}{2} \left.\frac{\partial^2 V}{\partial x^2}\right|_0 x^2$$

$$= \frac{1}{2} k x^2$$

LECTURE 18 JULY 23, 2007

SOLVE SHO DIFF EQ

$$H \,|\varphi_n\rangle = E_n \,|\varphi_n\rangle$$

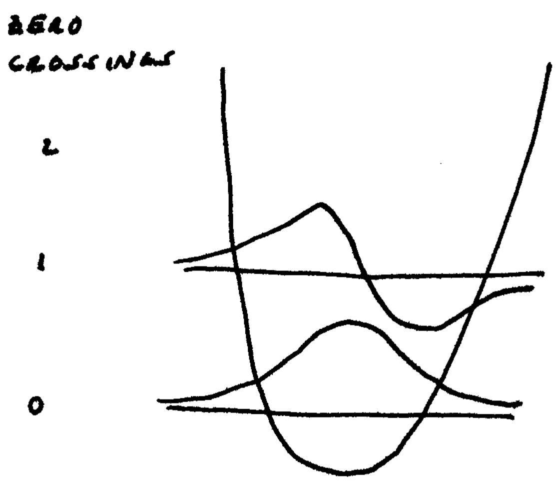
to get x-space, dot with $\langle x|$ bra

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right]\varphi_n(x) = E_n\,\varphi_n(x)$$

Find ev's E_n

Find ef's $\varphi_n(x)$

TELL 'EM WHAT YOU'RE GONNA TELL 'EM



FIND ASYPTOTIC FORM OUTSIDE

= GAUSSIAN

SOLUTION = GAUSSIAN X POLYNOMIALS

HERMITE POLYNOMIALS

TODAY: SHO-2 POWER SERIES,
ORTHOGONAL POLYNOMIALS,
DIFFERENTIAL EQUATIONS,
AND ALL THAT....

WE WANT TO SOLVE TDSE

$$H\,|\psi\rangle = i\hbar\,\frac{d}{dt}\,|\psi\rangle$$

SO WE FIRST SOLVE TISE

$$H\,|E_m\rangle = E_m\,|E_m\rangle$$

THEN USE

$$|\psi(t)\rangle = \sum_m |E_m\rangle\langle E_m|\; e^{-iE_m t/\hbar}\;|\psi(0)\rangle$$

$$= \sum_m |E_m\rangle\langle E_m|\psi(0)\rangle\; e^{-iE_m t/\hbar}$$

SO, TO SOLVE TISE...

7-26-06

LECTURE 11: SHO II

POWER SERIES, DIFF EQ'S, AND ALL THAT

TISE

$$H \, | \varphi_n \rangle = E_n \, | \varphi_n \rangle$$

using position basis:

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} K x^2 \right] \varphi_n(x) = E_n \, \varphi_n(x)$$

OUR JOB: FIND φ_n'S

FIND E_n'S

REWRITE TISE

$$\frac{d^2 \psi}{d x^2} + \left(\lambda - \alpha^2 x^2 \right) \psi = 0$$

$$\lambda = \frac{2 m E}{\hbar^2}$$

$$\alpha = \frac{m K}{\hbar^2}$$

FIRST STEP: FIND ASYMPTOTIC SOLUTION

FOR LARGE x, $\alpha^2 x^2 >> \lambda$

$$\frac{d^2\psi}{dx^2} - \alpha^2 x^2 \psi = 0$$

$$\psi = A e^{-\frac{1}{2}\alpha x^2} + B e^{+\frac{1}{2}\alpha x^2}$$

$$\frac{d\psi}{dx} = -\alpha x \, A e^{-1/2\,\alpha x^2} + \alpha x \, B e^{+1/2\,\alpha x^2}$$

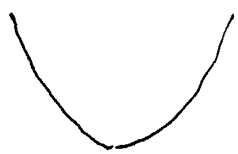
$$\frac{d^2\psi}{dx^2} = \alpha^2 x^2 \, A e^{-1/2\,\alpha x^2} + \alpha^2 x^2 \, B e^{+1/2\,\alpha x^2}$$

$$\cancel{-\alpha A e^{-1/2\,\alpha x^2}} \cancel{+\alpha B e^{+1/2\,\alpha x^2}}$$

$\alpha^2 x^2 >> \alpha$

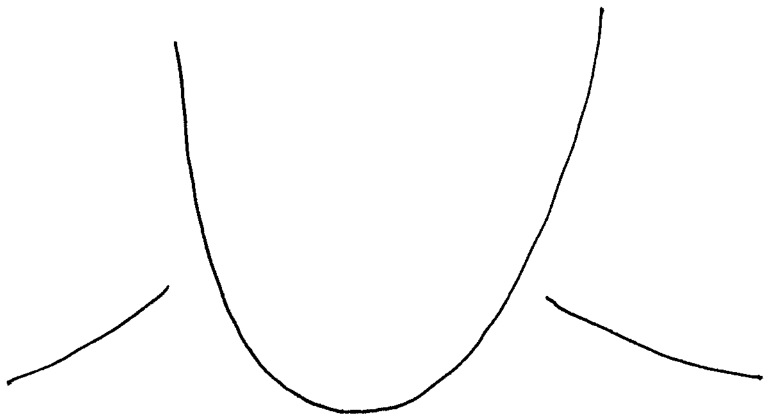
$$\psi(x) = A e^{-\frac{1}{2}\alpha x^2}$$

$$\psi(x) = B e^{+1/2\,\alpha x^2}$$



not square integ

SO, AT LARGE x, WE HAVE GAUSSIAN DECAY



LOOK FOR SOLUTIONS OF THE FORM

$$\psi(x) = e^{-\frac{1}{2}\alpha x^2} f(x)$$

$$\frac{d^2\psi}{dx^2} + (\lambda - \alpha^2 x^2)\,\psi = 0$$

$$\frac{d\psi}{dx} = \frac{d}{dx}\left[e^{-\frac{1}{2}\alpha x^2} f(x)\right]$$

$$= -\alpha x\, e^{-\frac{1}{2}\alpha x^2} f(x) + e^{-\frac{1}{2}\alpha x^2} \frac{df}{dx}$$

$$\frac{d^2\psi}{dx^2} = -\alpha e^{-\frac{1}{2}\alpha x^2} f(x) + \alpha^2 x^2 e^{-1/2\,\alpha x^2} f(x)$$

$$-\alpha x\, e^{-1/2\,\alpha x^2} \frac{df}{dx} - \alpha x\, e^{-1/2\,\alpha x^2} \frac{df}{dx}$$

$$+\, e^{-\frac{1}{2}\alpha x^2} \frac{d^2 f}{dx^2}$$

$$\frac{d^2\psi}{dx^2} \Rightarrow \left[\frac{d^2 f}{dx^2} - 2\alpha x \frac{df}{dx} + (\cancel{\alpha^2 x^2} - \alpha) f \right] e^{-1/2\,\alpha x^2}$$

$$+ (\lambda - \alpha^2 x^2)\psi = 0$$

$$+ (\lambda - \cancel{\alpha^2 x^2}) f\, e^{-1/2\,\alpha x^2} = 0$$

$$0 = \left[\frac{d^2 f}{dx^2} - 2\alpha x \frac{df}{dx} + (\lambda - a) f \right] \cancel{e^{-1/2\,\alpha x^2}}$$

$$\xi = \sqrt{\alpha}\, x$$

$$f(x) \longrightarrow H(\xi)$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} H(\xi) = \frac{d}{d\xi} H(\xi) \frac{d\xi}{dx}$$

$$= \frac{dH(\xi)}{d\xi} \sqrt{\alpha}$$

$$\frac{d^2 f}{dx^2} = \alpha \frac{d^2 H}{d\xi^2}$$

$$\frac{d^2 f}{dx^2} - 2\alpha x \frac{df}{dx} + (\lambda - \alpha) f = 0$$

$$\frac{1}{\alpha} \left[\alpha \frac{d^2 H}{d\xi^2} - 2\alpha x \frac{dH}{d\xi} \sqrt{\alpha} + (\lambda - \alpha) H \right] = 0$$

$$\xi = \sqrt{\alpha}\, x$$

$$\boxed{\frac{d^2 H}{d\xi^2} - 2\xi \frac{dH}{d\xi} + \left(\frac{\lambda}{\alpha} - 1\right) H = 0}$$

HERMITE'S EQN

SOLVE USING POWER SERIES ...

$$H(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + \cdots$$

$$\frac{dH}{d\xi} = 0 + a_1 + 2 a_2 \xi + 3 a_3 \xi^2 + \cdots$$

$$\frac{d^2H}{d\xi^2} = 0 + 0 + 1\cdot 2\, a_2 \cancel{\xi} + 2\cdot 3\, a_3 \xi$$

KNOW

$$\frac{d^2H}{d\xi^2} - 2\xi \frac{dH}{d\xi} + \left(\frac{\lambda}{\alpha} - 1\right) H = 0 \qquad \text{FOR ALL } \xi \, !$$

$\Rightarrow$ each coefficient of ξ must vanish

ξ^0 $\qquad 1\cdot 2\, a_2 + \left(\frac{\lambda}{\alpha} - 1\right) a_0 = 0$

ξ^1 $\qquad 2\cdot 3\, a_3 + \left(\frac{\lambda}{\alpha} - 1 - 2\right) a_1 = 0$

ξ^2 $\qquad 3\cdot 4\, a_4 + \left(\frac{\lambda}{\alpha} - 1 - 2\cdot 2\right) a_2 = 0$

ξ^3 $\qquad 4\cdot 5\, a_5 + \left(\frac{\lambda}{\alpha} - 1 - 2\cdot 3\right) a_3 = 0$

$\Rightarrow$ COEFF OF ξ^m

$$(m+1)(m+2)\, a_{m+2} + \left(\frac{\lambda}{\alpha} - 1 - 2m\right) a_m = 0$$

$$\boxed{a_{m+2} = \frac{-\left(\frac{\lambda}{\alpha} - 2m - 1\right)}{(m+2)(m+1)}\, a_m}$$

RECURSION RELATION

$\Delta m = 2$ $\qquad a_0 \Rightarrow$ all even COEFF'S

$\qquad a_1 \Rightarrow$ all ODD COEFF'S

FOR EACH m:

WHEN $\left(+\frac{\lambda}{\alpha} - 2m - 1\right) = 0$

all higher terms vanish!

SPECIAL VALUES OF $\lambda \Rightarrow$ FINITE POLYNOMIALS

$$\lambda = \frac{2mE}{\hbar}$$

$\Rightarrow$ EIGENENERGIES !

TWO CLASSES OF SOLUTIONS

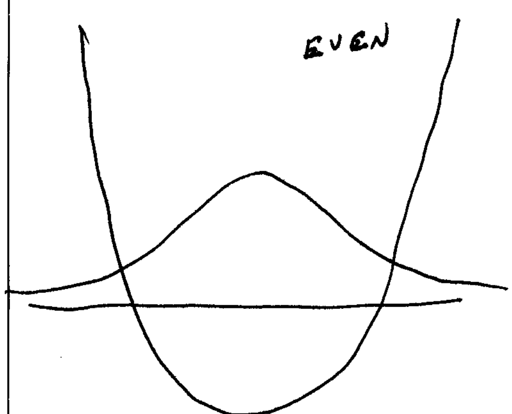
EVEN

$$H(\xi) = a_0 + a_2 \xi^2 + a_4 \xi^4 + \ldots$$

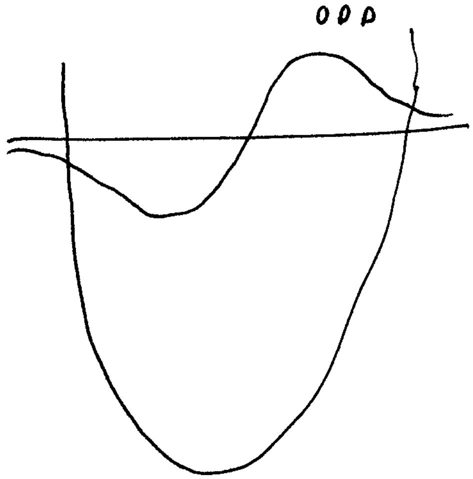
ODD

$$H(\xi) = a_1 \xi + a_3 \xi^3 + a_5 \xi^5 + \ldots$$

ODD AND EVEN COME FROM SYMMETRY OF THE PROBLEM



ODD



N. B., FOR AN ARBITRARY ENERGY, NO STATIONARY STATE! SERIES ~~DO~~ DOES NOT TERMINATE!

$$a_{m+2} = \frac{-\left[\frac{\lambda}{\alpha} - 2m - 1\right]}{(m+2)(m+1)}\, a_m$$

CAN CHOOSE λ TO TERMINATE EVEN OR ODD SERIES, BUT NOT BOTH.

LARGE m

$$a_{m+2} = \frac{-\left[\cancel{\frac{\lambda}{\alpha}} - 2m - \cancel{1}\right]}{(m+\cancel{2})(m+\cancel{1})}\, a_m$$

$$a_{m+2} = +\left(\frac{2}{m}\right) a_m$$

COMPARE WITH POWER SERIES FOR A GAUSSIAN

$$e^{\xi^2} = \sum_{n=0}^{\infty} \frac{(\xi^2)^n}{n!} = 1 + \xi^2 + \frac{1}{2!}\xi^4 + \frac{1}{3!}\xi^6$$

$$+ \frac{\xi^m}{\left(\frac{m}{2}\right)!} + \frac{\xi^{m+2}}{\left(\frac{m+2}{2}\right)!}$$

$$\frac{b_{m+2}}{b_m} = \left(\frac{2}{m}\right) \Rightarrow \text{SAME!}$$

CALCULATE THE EIGENENERGIES

$$\left[\frac{\lambda}{\alpha} - 2m - 1 \right] = 0$$

$$\frac{\lambda}{\alpha} = 2m + 1$$

$$\lambda = (2m+1)\,\alpha$$

$$\frac{2mE}{\hbar^2} = (2m+1)\sqrt{\frac{mK}{\hbar^2}}$$

$$E = (2m+1)\,\frac{\hbar^2}{2m}\sqrt{\frac{mK}{\hbar^2}}$$

$$= (m + 1/2)\,\hbar\sqrt{\frac{K}{m}}$$

$$= (m + 1/2)\,\hbar\,\omega$$

$$
\begin{aligned}
H_0(\xi) &= 1 \\
H_1(\xi) &= 2\xi \\
H_2(\xi) &= 4\xi^2 - 2 \\
H_3(\xi) &= 8\xi^3 - 12\xi \\
H_4(\xi) &= 16\xi^4 - 48\xi^2 + 12 \\
H_5(\xi) &= 32\xi^5 - 160\xi^3 + 120\xi \\
H_6(\xi) &= 64\xi^6 - 480\xi^4 + 720\xi^2 - 120 \\
H_7(\xi) &= 128\xi^7 - 1344\xi^5 + 3360\xi^3 - 1680\xi \\
H_8(\xi) &= 256\xi^8 - 3584\xi^6 + 13440\xi^4 - 13440\xi^2 + 1680 \\
H_9(\xi) &= 512\xi^9 - 9216\xi^7 + 48384\xi^5 - 80640\xi^3 + 30240\xi \\
H_{10}(\xi) &= 1024\xi^{10} - 23040\xi^8 + 161280\xi^6 - 403200\xi^4 + 302400\xi^2 \\
&\qquad - 30240.
\end{aligned}
\tag{11–23}
$$

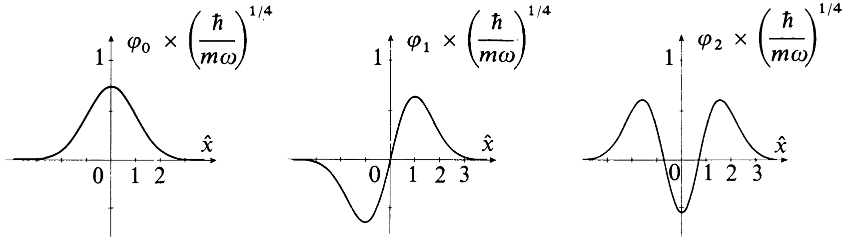


FIGURE 4

Wave functions associated with the first three levels of a harmonic oscillator.

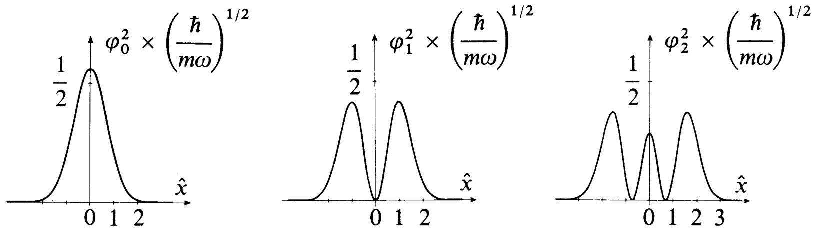


FIGURE 5

Probability densities associated with the first three levels of a harmonic oscillator.

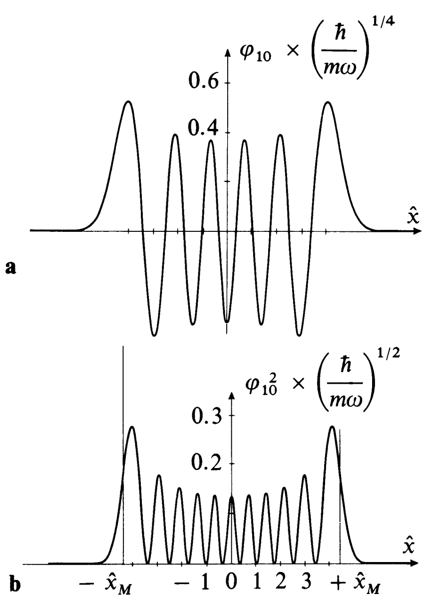


FIGURE 6

Shape of the wave function (fig. a) and of the probability density (fig. b) for the $n = 10$ level of a harmonic oscillator.

The Harmonic Oscillator

http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/hosc.html
http://www.falstad.com/qm1d/
http://www-personal.umich.edu/~lorenzon/java_applets/spaceholder/applets/SHO-QM-example.html?D1=5
http://www.quantum-physics.polytechnique.fr/en/

Hermite Polynomials

http://mathworld.wolfram.com/HermitePolynomial.html
http://www.efunda.com/math/Hermite/index.cfm
http://www.sci.wsu.edu/idea/quantum/hermite.htm
http://functions.wolfram.com/Polynomials/

Coherent States

http://cat.sckans.edu/physics/Quantum%20Wave%20Ppacket.htm