LECTURE 21 7-29-09

FREE PARTICLE

$$\psi_+(x,t) = A e^{ikx} e^{-i\omega t} = A e^{i(kx - \omega t)}$$

$$= A e^{ikx} e^{-iEt/\hbar}$$

$$= A e^{ikx} e^{-i\left(\frac{\hbar^2 k^2}{2m}\right) t/\hbar}$$

$$A e^{ipx/\hbar} \exp\left(-i \left(\frac{p^2}{2m}\right) t/\hbar\right)$$

$f(x - v t)$ WAVE

$f\left(x - \frac{\omega}{k} t\right)$

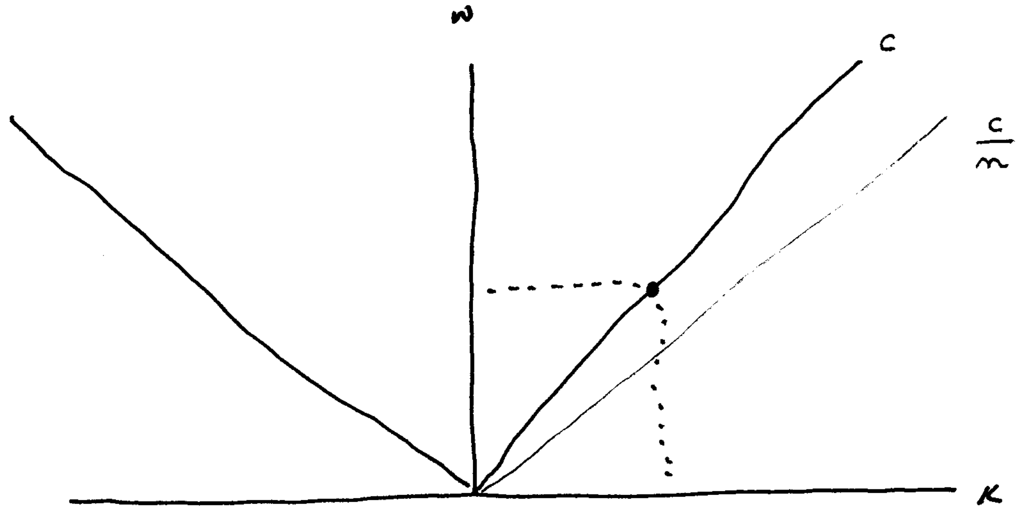
LIGHT $\omega = ck \Rightarrow v = c$

MATTER $E = \hbar\omega = \dfrac{\hbar^2 k^2}{2m}$

$$\omega = \left(\frac{\hbar}{2m}\right) k^2$$

$$v = \frac{\omega}{k} = \frac{1}{2} \frac{\hbar k}{m} = \frac{1}{2} \frac{p}{m} = \frac{1}{2} v_{\text{CLASSICAL}}$$

LIGHT IN VACUUM




PHASE VELOCITY $v_p = \frac{\omega}{K}$ = slope of chord from (0,0) to (K, ω)

$v_p = c$ in vacuum

$v_p = \frac{c}{n}$ in matter

GROUP VELOCITY $v_g = \frac{d\omega}{dK} = c$ in vacuum

$v_g = \frac{c}{n(\omega)}$ frequency dependent

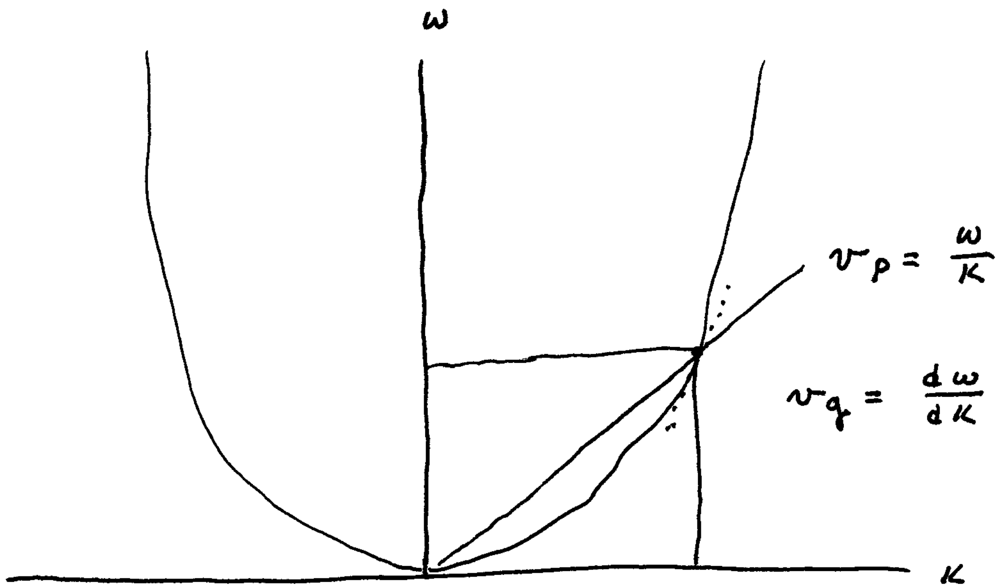
color dependent

PRISM

RAINBOW

PEACOCKS

⋮




$$E = \hbar\omega = \frac{\hbar^2 K^2}{2m} \qquad \omega = \left(\frac{\hbar}{2m}\right) K^2$$

$$v_p = \frac{\omega}{K} = \frac{\left(\frac{\hbar}{2m}\right) K^2}{K}$$

$$= \frac{1}{2}\frac{\hbar K}{m} = \frac{1}{2}\frac{p}{m}$$

$$v_p = \frac{1}{2} v_{cl}$$

$$v_g = \frac{d\omega}{dK} = \frac{d}{dK}\left(\frac{\hbar}{2m} K^2\right)$$

$$v_g = \frac{\hbar K}{m} = \frac{p}{m}$$

$$v_g = v_{cl}$$

$v_g = 2 v_p$ FOR MATTER WAVES

NOW

$e^{ikx} \quad e^{-i\omega t} \qquad \infty \text{ in } x$

$\infty \text{ in } t$

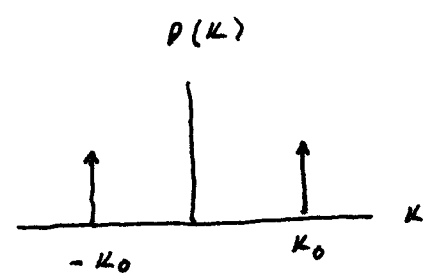
$\Delta x \, \Delta p \geq \frac{\hbar}{2} \qquad \Delta x \to \infty \quad \Delta p = 0 \qquad \Delta k = 0$

$\Delta E \Delta t \geq \frac{\hbar}{2} \qquad \Delta t \to \infty$

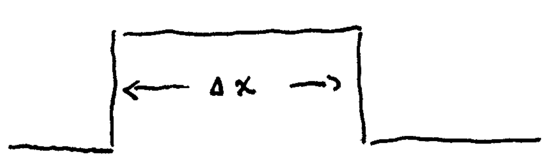
$\Delta E \to 0$

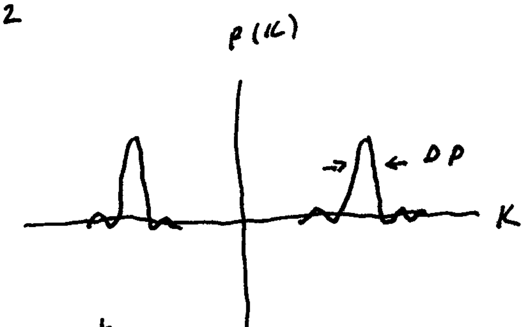
PURE es of momentum must be ∞ long in space

energy time



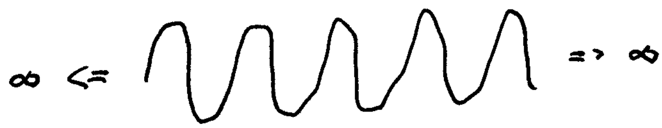

$\cos(kx) = \left(e^{ikx} + e^{-ikx}\right)\frac{1}{2}$



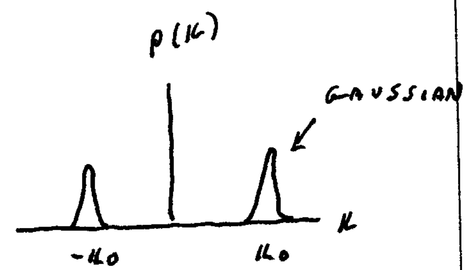




$\Delta x \, \Delta p \geq \frac{\hbar}{2}$

FINITE EXTENT => MOMENTUM SPREAD









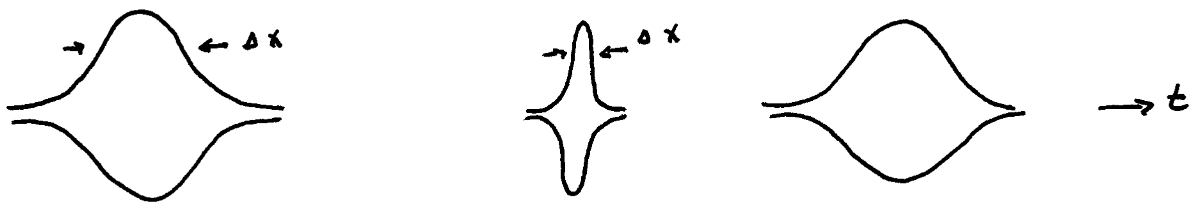
=> SHAPE CHANGES!

HOW DOES IT CHANGE?

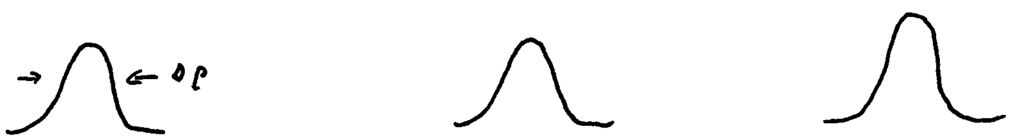
CHANGES IN X

CONSTANT IN P

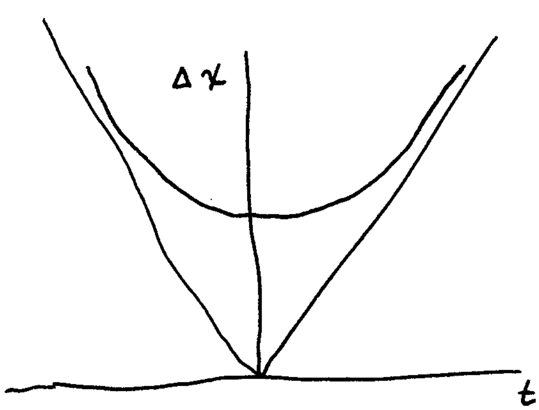
GENERAL SOLUTION

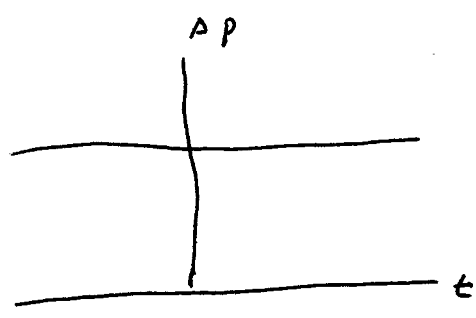



SYMMETRIC IN TIME




$\Delta p$ INDEPENDENT OF TIME



$$\psi(x,0) \sim e^{-x^2/2\Delta^2}$$

GAUSSIAN ENVELOPE

MIN WIDTH => $\Delta X = \frac{\Delta}{\sqrt{2}}$

TIME-DEPENDENT WIDTH

$$\Delta x(t) = \frac{\Delta}{\sqrt{2}} \sqrt{1 + \frac{\hbar^2 t^2}{m^2 \Delta^4}}$$

AT $t = 0$ $\quad \Delta x = \frac{\Delta}{\sqrt{2}}$

WHEN $\frac{\hbar^2 t^2}{m^2 \Delta^4} \gg 1$ $\quad \Delta x(t) = \left(\frac{\Delta}{\sqrt{2}} \frac{\hbar}{m\Delta^2}\right) t$

$\sim \frac{t}{\Delta}$ SMALLER Δ => FASTER SPREAD

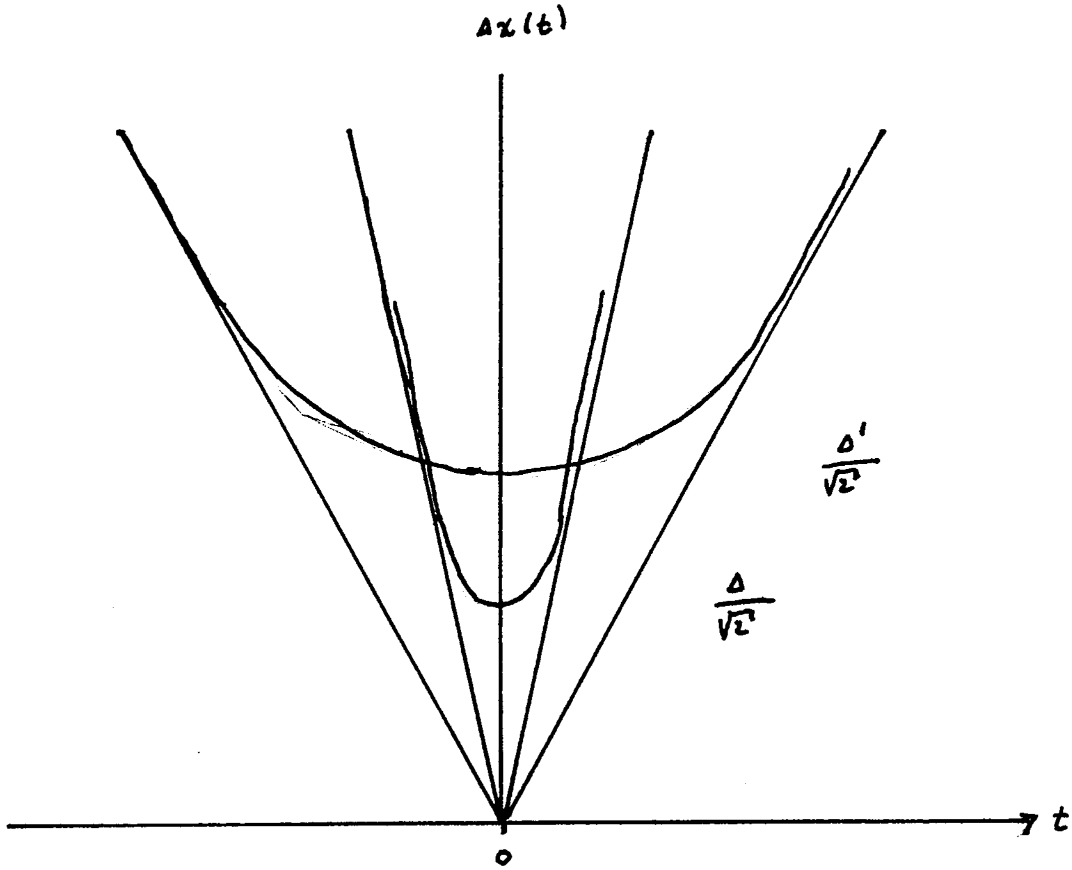
$\Delta p(t) = \frac{\hbar}{\sqrt{2}\,\Delta}$ CONSTANT

$\Delta v = \frac{\hbar}{\sqrt{2}\, m \Delta}$ SMALLER Δ => LARGE Δv

CLASSICAL SPREAD

$\Delta x(t) = \Delta v\, t$

NOT TO SCALE!



Δx(t)
Δ'/√2
Δ/√2
t
0

# Pictures of Water Waves You Know and Love

# Duck wake

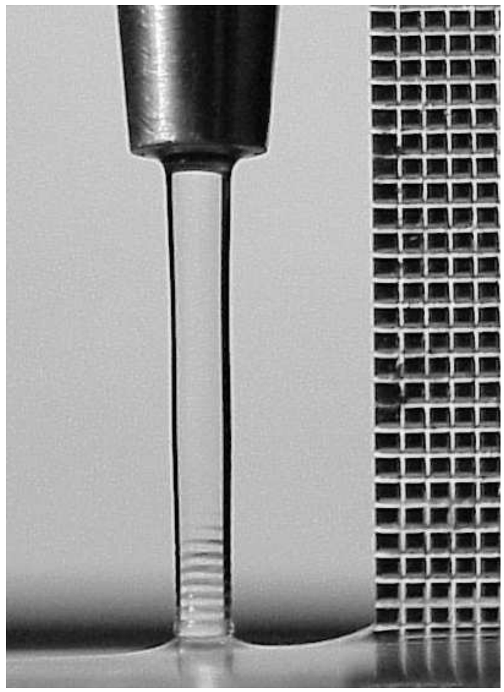


Figure 5: The field of stationary capillary waves excited on the base of a water jet impinging on a horizontal water reservoir. The grid at right is millimetric.

Cancellation via (9) yields the equation for $\widetilde{p}$ accurate to order $\epsilon$:

$$\widetilde{p} = -\frac{\epsilon\sigma}{R_0^2}\left(1 - k^2 R_0^2\right) e^{\omega t + ikz} \quad . \tag{27}$$

Combining (20), (22) and (27) yields the dispersion relation, that indicates the dependence of the growth rate $\omega$ on the wavenumber $k$:

$$\boxed{\omega^2 = \frac{\sigma}{\rho R_0^3} k R_0 \frac{I_1(kR_0)}{I_0(kR_0)} \left(1 - k^2 R_0^2\right)} \quad . \tag{28}$$

We first note that unstable modes are only possible when

$$kR_0 < 1 \tag{29}$$

The column is thus unstable to disturbances whose wavelengths exceed the circumference of the cylinder. A plot for the dispersion relation is shown in Figure 4.

The fastest growing mode occurs for $kR_0 = 0.697$, i.e. when the wavelength of the disturbance is

$$\lambda_{max} \simeq 9.02 R_0 \quad . \tag{30}$$

By inverting the maximum growth rate $\omega_{max}$ one may estimate the characteristic break up time:

$$t_{breakup} \simeq 2.91 \sqrt{\frac{\rho R_0^3}{\sigma}} \quad . \tag{31}$$

A water jet of diameter 1cm has a characteristic break-up time of about 1/8 s, which is consistent with casual observation of jet break-up in a kitchen sink.

When a vertical water jet impinges on a horizontal reservoir of water, a field of standing waves may be excited on the base of the jet (see Figure 5). The wavelength is determined by the requirement that the wave speed correspond to the local jet speed: $U = -\omega/k$. Using our dispersion relation (28) thus yields

$$U^2 \;=\; \frac{\omega^2}{k^2} \;=\; \frac{\sigma}{\rho k R_0^2} \frac{I_1(kR_0)}{I_0(kR_0)} \left(1 - k^2 R_0^2\right) \quad . \tag{32}$$

Provided the jet speed $U$ is known, this equation may be solved in order to deduce the wavelength of the waves that will travel at $U$ and so appear to be stationary in the lab frame. For jets falling from a nozzle, the result (4) may be used to deduce the local jet speed.

**5.3 Fluid Pipes** (see http://www-math.mit.edu/ bush/pipes.html)

The following system may be readily observed in a kitchen sink. When the volume flux exiting the tap is such that the falling stream has a diameter of 2-3mm, obstructing the stream with a finger at a distance of several centimeters from the tap gives rise to a stationary field of varicose capillary waves upstream of the finger. If the finger is dipped in liquid detergent (soap) before insertion into the stream, the capillary waves begin at some critical distance above the finger, below which the stream is cylindrical. Closer inspection reveals that the surface of the jet's cylindrical base is quiescent.

An analogous phenomenon arises when a vertical fluid jet impinges on a deep water reservoir (Figures 5 and 6). When the reservoir is contaminated by surfactant, the surface tension of the reservoir is diminished relative to that of the jet. The associated surface tension gradient draws surfactant a finite distance up the jet, prompting two salient alterations in the jet surface. First, the surfactant suppresses surface waves, so that the base of the jet surface assumes a cylindrical form (Figure 6). Second, the jet surface at its base becomes stagnant: the Marangoni stresses associated with the surfactant gradient are balanced by the viscous stresses generated within the jet. The quiescence of the jet surface may be simply demonstrated by sprinkling a small amount of talc or lycopodium powder onto the jet. The fluid jet thus enters a contaminated reservoir as if through a rigid pipe.

A detailed theoretical description of the fluid pipe is given in Hancock & Bush (JFM, **466**, 285-304). We here present a simple scaling that yields the dependence of the vertical extent $H$ of the fluid pipe on the governing system parameters. We assume that, once the jet enters the fluid pipe, a boundary layer develops on its outer wall owing to the no-slip boundary condition appropriate there. Balancing viscous and Marangoni stresses on the pipe surface yields

$$\rho\nu \frac{V}{\delta_H} \;\sim\; \frac{\Delta\sigma}{H} \quad , \tag{33}$$

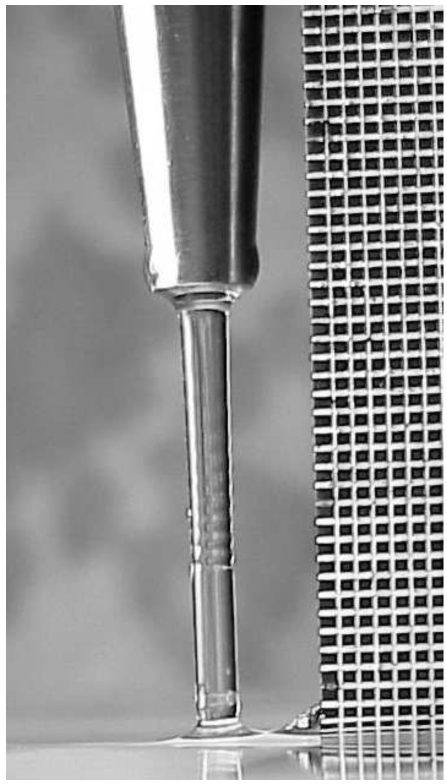


Figure 6: The fluid pipe generated by a falling water jet impinging on a contaminated water reservoir. The field of stationary capillary waves is excited above the fluid pipe. The grid at right is millimetric.

where $\Delta\sigma$ is the surface tension differential between the jet and reservoir, $V$ is the jet speed at the top of the fluid pipe, and $\delta_H$ is the boundary layer thickness at the base of the fluid pipe. We assume that the boundary layer thickness increases with distance $z$ from the inlet according to classical boundary layer scaling:

$$\frac{\delta}{a} \sim \left(\frac{\nu z}{a^2 V}\right)^{1/2} \quad . \tag{34}$$

Substituting for $\delta(H)$ from (34) into (33) yields

$$H \sim \frac{(\Delta\sigma)^2}{\rho\mu V^3} \quad . \tag{35}$$

The pipe height increases with the surface tension differential and pipe radius, and decreases with fluid viscosity and jet speed.

# Ships

# Dispersion relation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

assume: $u(x,t) = Ae^{i(kx-\omega t)}$

$$\omega^2 = c^2 k^2$$

Phase velocity: $\frac{\omega}{\mathrm{k}} = c$

Group velocity: $\frac{\mathrm{d}\omega(k)}{\mathrm{dk}} = \mathrm{c}$

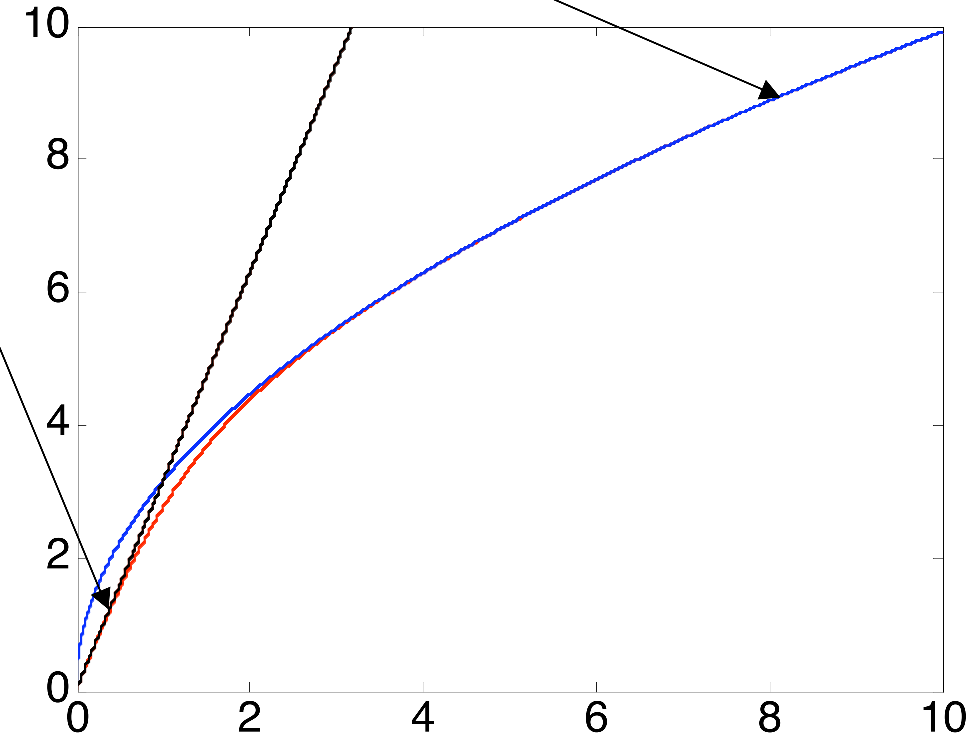
The “regular” wave equation is non-dispersive

# Shallow and deep water

$$\omega^2 = gk\tanh(kH)$$

$\omega^2 = gk$ Deep water approx.

$\omega^2 = gHk^2$ Shallow water approx.



# Shallow water

$$\omega^2 = gHk^2$$

$$\text{Phase velocity:} \quad \frac{\omega}{\mathrm{k}} = \sqrt{gH}$$

$$\text{Group velocity:} \quad \frac{\mathrm{d}\omega(k)}{\mathrm{dk}} = \sqrt{gH}$$

Shallow water waves are “ordinary” waves when amplitude is small
Shallow water wave speed depends on height of wave – results in breaking
Shallow water waves are hyperbolic

# Deep water waves

$$\omega^2 = gk$$

Phase velocity: $\dfrac{\omega}{\mathrm{k}} = \sqrt{\dfrac{g}{k}}$

Group velocity: $\dfrac{\mathrm{d}\omega(k)}{\mathrm{dk}} = \dfrac{1}{2}\sqrt{\dfrac{g}{k}}$

Deep water waves are dispersive.
Long waves are faster than short ones.
Group velocity is ½ of the phase velocity.

$$\Omega^2(k) \;=\; \left(g \;+\; \frac{\gamma}{\rho}\,k^2\right)\; k\;\; \tanh\,(k\,h),$$

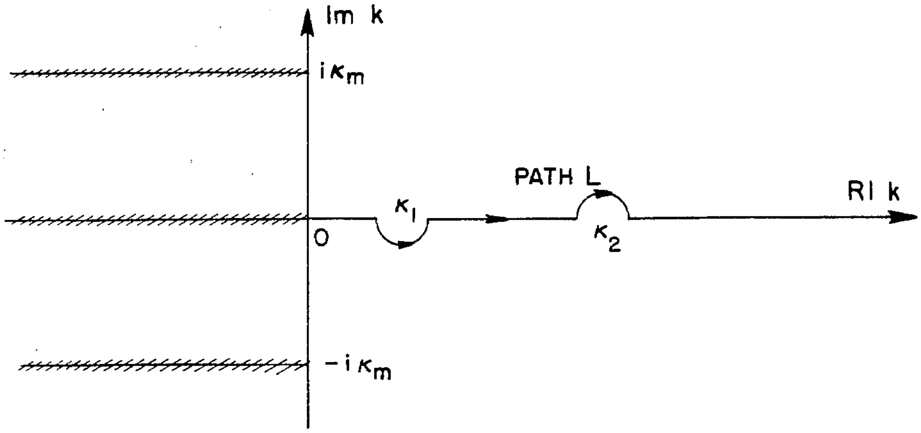



Fig. 1

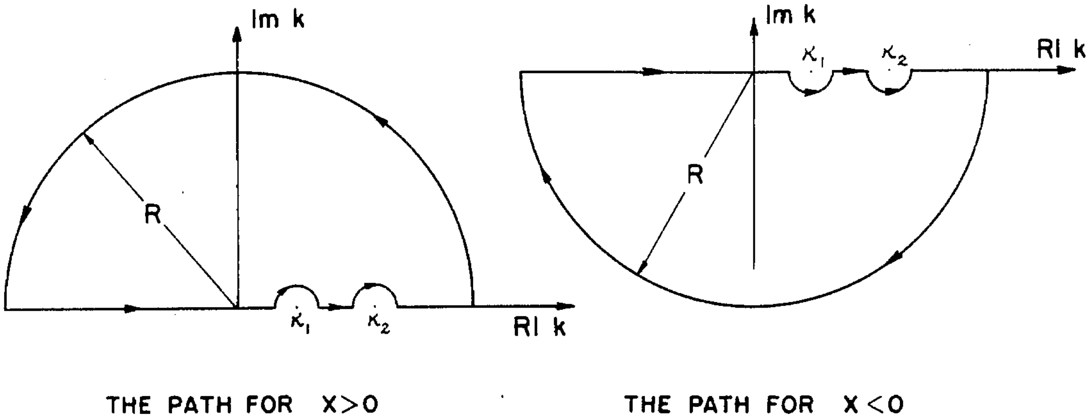



Fig. 2

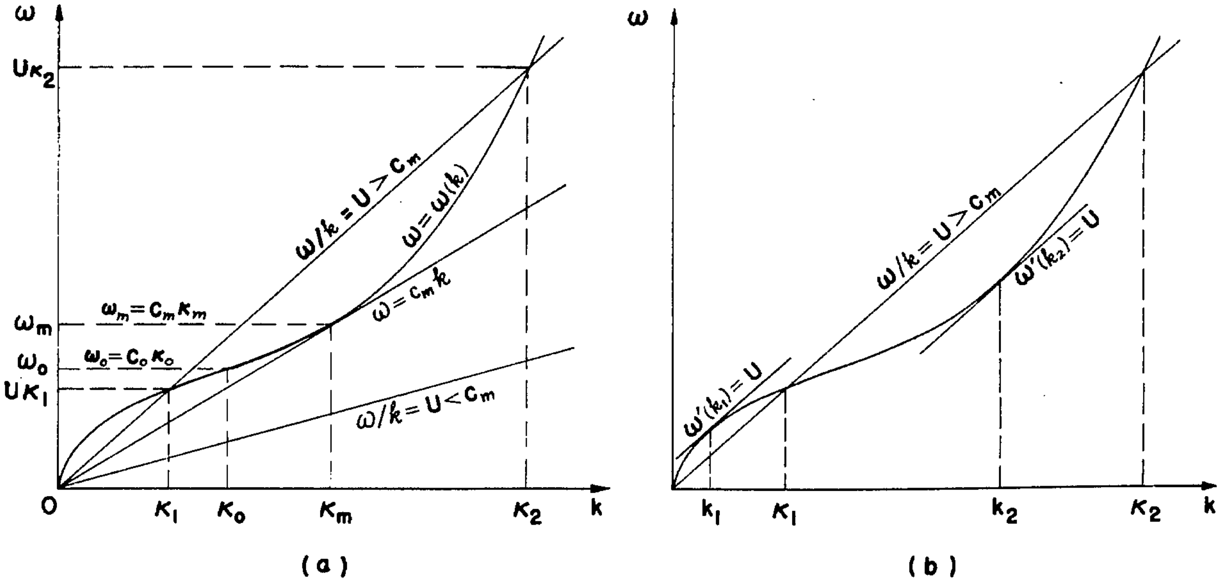



Fig. 3. Qualitative features of the curve $\omega = \omega(k)$.

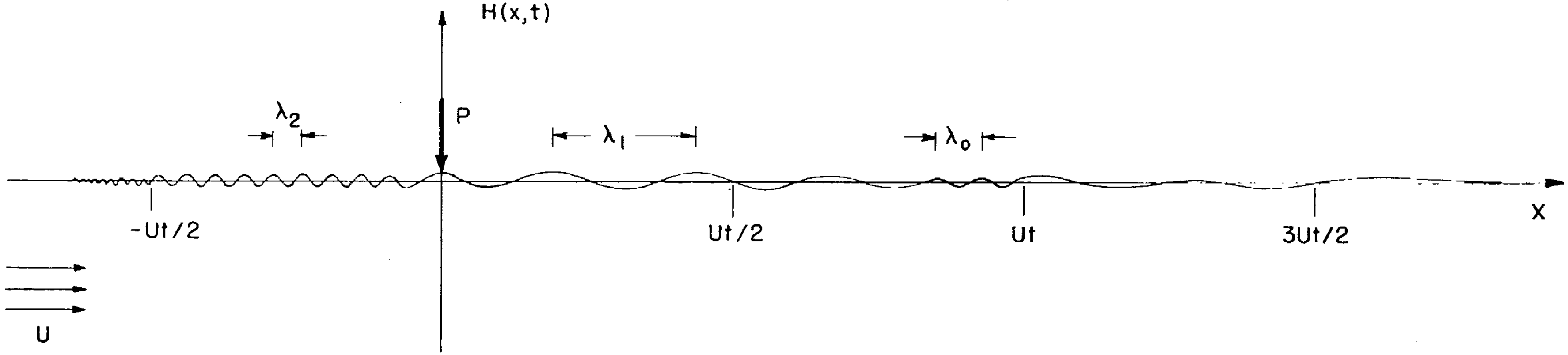
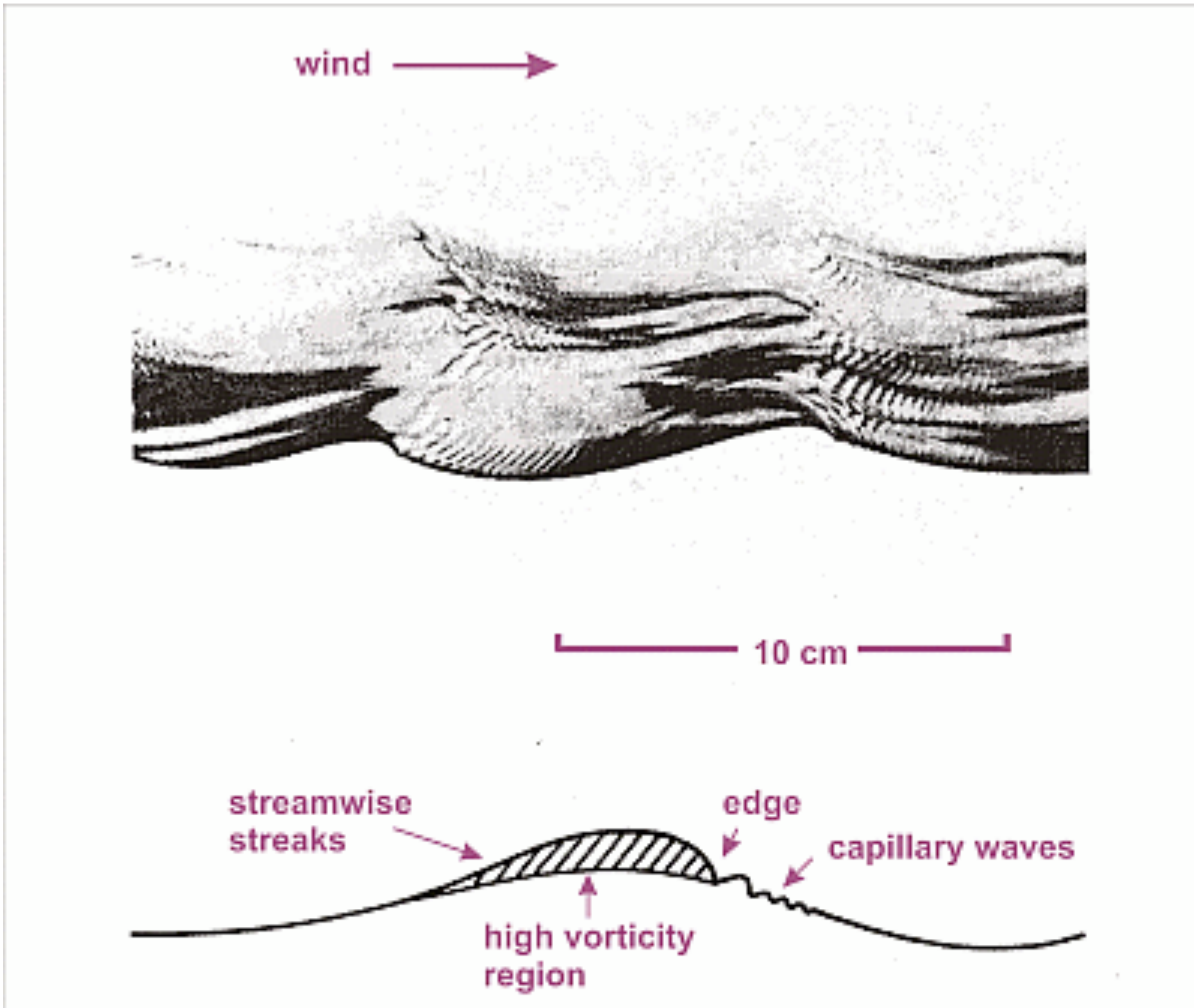



Fig. 4. The space distribution of the wave trains for large time.


wind
10 cm
streamwise
streaks
edge
capillary waves
high vorticity
region