Virtual Book

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LECTURE 11 JULY 10, 2007

TODAY: SOLVE FREE PARTICLE PROBLEM

TWO WAYS

(1) ABSTRACT METHOD

$$\mathcal{H} = \frac{p^2}{2m} + V(x)$$

FREE $\Rightarrow$ $F = 0$ $\quad F = \frac{\partial V}{\partial x}$ $\Rightarrow$ $V =$ CONSTANT $= 0$

$$H = \frac{p_{op}^2}{2m}$$

$$H\,|E_n\rangle = E_n\,|E_n\rangle \qquad \text{TISE}$$

$$\frac{p_{op}^2}{2m}\,|E_n\rangle = E_n\,|E_n\rangle$$

$[H, p_{op}] = 0$ $\Rightarrow$ H AND p_{op} SHARE ALL $\vec{ev}$'S

$$\frac{p_{op}^2}{2m}\,|p\rangle = E\,|p\rangle$$

$$p_{op}\,|p\rangle = p\,|p\rangle \Rightarrow \frac{p^2}{2m}\,|p\rangle = E\,|p\rangle$$

$$\Rightarrow \frac{p^2}{2m} = E$$

$$p = \pm\sqrt{2mE}$$

AKA, OPERATOR METHOD

BASIS FREE

ALGEBRA OF THE OPERATORS

$\Rightarrow$ TWO ORTHONORMAL EIGENKETS FOR EACH EIGEN ENERGY.

H_{op} IS 2-FOLD DEGENERATE

P_{op} IS NOT $\quad |+p\rangle$ momentum eigen KETS

$\qquad\qquad\qquad\quad |-p\rangle$

$|E, p\rangle$

$|E,+\rangle = |E, p = +\sqrt{2mE}\rangle$ energy eigen KETS

$|E,-\rangle = |E, p - \sqrt{2mE}\rangle$

SUPERPOSITION STATES

$|\psi\rangle = \alpha\,|E,+\rangle + \beta\,|E,-\rangle$ also energy eigen KETS

MEASURE H, OBTAIN E 100% OF THE TIME

MEASURE p ?

$$\text{PROB}\left(p = +\sqrt{2mE}\right) = \frac{|\langle +p|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}$$

$$\text{PROB}\left(p = -\sqrt{2mE}\right) = \frac{|\langle -p|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}$$

IF $|\psi\rangle$ IS NORMALIZED $\quad \langle\psi|\psi\rangle = 1 = |\alpha|^2 + |\beta|^2$

$$|\psi(t)\rangle = |E,+\rangle\langle E,+|\psi(0)\rangle\, e^{-iEt/\hbar}$$
$$+\, |E,-\rangle\langle E,-|\psi(0)\rangle\, e^{-iEt/\hbar}$$

(2) DIFF EQ METHOD (TISE)

$$H \mid E_n \rangle = E_n \mid E_n \rangle$$

in-X SPACE

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) = E_n \psi_n(x) \quad \text{TISE}$$

TRIAL SOLUTION

$$\psi_n(x) = A e^{ikx} + B e^{-ikx}$$

$$-\frac{\hbar^2}{2m} (\pm i k)^2 \psi_n(x) = E_n \psi_n(x)$$

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}$$

(2') DIFF EQ METHOD TDSE

IN HILBERT SPACE: $H\,|\psi(t)\rangle = i\hbar \dfrac{d}{dt}\,|\psi(t)\rangle$

IN POSITION SPACE:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\,\psi(x,t) = i\hbar\frac{d}{dt}\,\psi(x,t)$$

SECOND ORDER IN SPACE

FIRST ORDER IN TIME

TRIAL SOLUTION $\psi(x,t) = A\,e^{i(kx-\omega t)} + B\,e^{-i(kx+\omega t)}$

$ikx - i\omega t \qquad -ikx - i\omega t$

$-iE_n t/\hbar \qquad -iE_n t/\hbar$

$$-\frac{\hbar^2}{2m}(\pm ik)^2\,\psi(x,t) = i\hbar(-i\omega)\,\psi(x,t)$$

$$\frac{\hbar^2 k^2}{2m} = \hbar\omega$$

$p = \hbar k$

$E = \hbar\omega$ $\quad \Rightarrow \quad \dfrac{p^2}{2m} = E$

DIFF EQ VIEW OF TDSE → TISE

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x,t) = i\hbar\frac{d}{dt}\psi(x,t)$$

ANSATZ $\quad \psi(x,t) = f(x)\,g(t)$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\big(f(x)\,g(t)\big) = i\hbar\frac{d}{dt}\big(f(x)\,g(t)\big)$$

$$g(t)\left(-\frac{\hbar^2}{2m}\right)\frac{d^2f}{dx^2} = \theta f(x)\,(i\hbar)\frac{dg}{dt}$$

$\frac{1}{f\cdot g}$

$$\frac{-\frac{\hbar^2}{2m}\frac{d^2f}{dx^2}}{f} = \frac{i\hbar\frac{dg}{dt}}{g}$$

LHS FCN ONLY OF X

RHS FCN ONLY OF t

HOW CAN THAT BE? = A CONSTANT = E_n

$$i\hbar\frac{dg}{dt} = E_n g$$

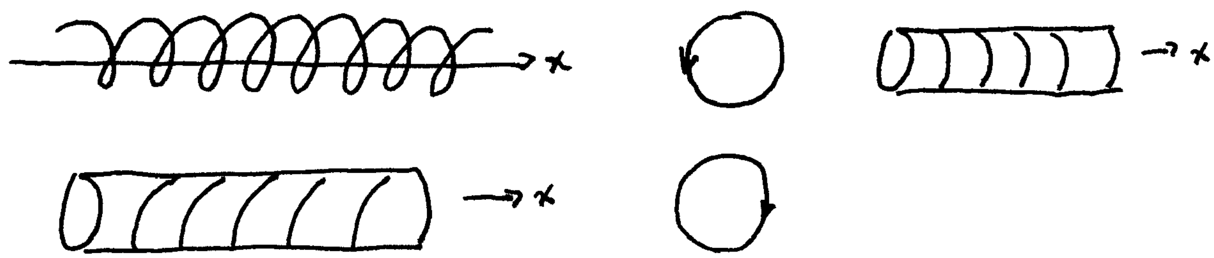
$$\frac{dg}{dt} = (-iE_n/\hbar)\,g \qquad g(t) = e^{-iE_nt/\hbar}$$

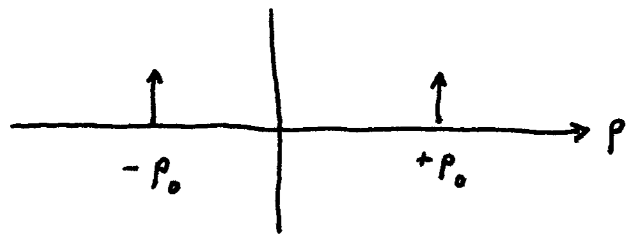
$\Rightarrow$ TISE $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} = E_n\psi_n$

MOMENTUM EIGENSTATES $\quad |p\rangle \rightarrow \langle x | p \rangle =$

$\psi(x)$

INFINITE EXTENT HELICAL WAVES





$\langle p | p_0 \rangle =$

$\hat{\psi}(p)$

ANY WAVE CAN BE EXPRESSED AS A LINEAR COMBINATION OF MOMENTUM EIGENSTATES

$$I = \int |p\rangle\langle p| \, dp$$

$$|\psi\rangle = I|\psi\rangle = \int dp \, |p\rangle\langle p|\psi\rangle$$

$\langle x|$ both sides $\Rightarrow$ go into the x basis

$$\langle x|\psi\rangle = \int dp \, \langle x|p\rangle\langle p|\psi\rangle$$

$$\psi(x) = \int dp \, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \, \tilde{\psi}(p)$$

OTHER WAY AROUND

$$I = \int dx \; |x\rangle\langle x|$$

dot with $\langle p|$

$$\langle p|\psi\rangle = \int dx \; \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \; \psi(x)$$

THE PROPAGATOR

$$U(t) = \int dp \; |p\rangle\langle p| \; e^{-iE(p)t/\hbar}$$

$$= \int dp \; e^{-ip^2 t/2m\hbar} \; |p\rangle\langle p|$$

WE WILL USE THIS FOR THE FREE PARTICLE

WAVE PACKET PROBLEM