LECTURE 12 7-13-09

GENERAL SOLNS TO $\nabla^2 V = 0$

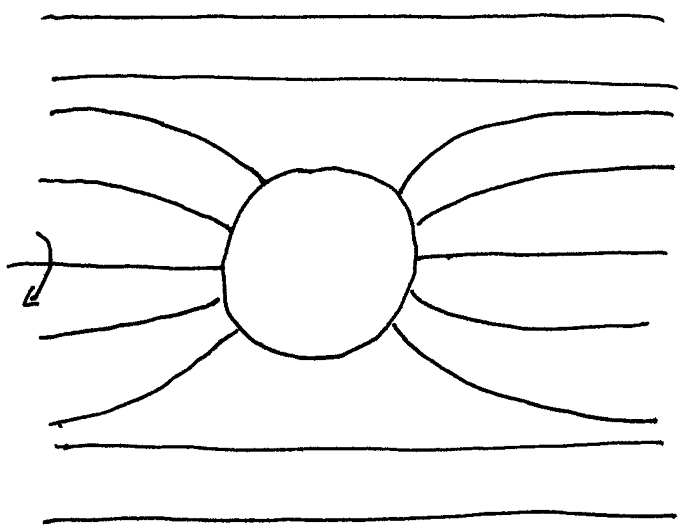
CARTESIAN

SPHERICAL

CYLINDRICAL

EXAMPLE PROBLEMS

① CONDUCTING SPHERE INTO A UNIFORM $\vec{E}$ FIELD



$$E(r,\theta,\varphi) = E_{APPLIED}(r,\theta,\varphi) + E_{INDUCED}(r,\theta,\varphi)$$

GEOMETRY

ORIGIN OF THE COORDS AT CENTER OF SPHERE

POLAR AXIS (z AXIS) ALONG FIELD

RADIUS OF SPHERE = a

SPHERE IS AN EQUIPOTENTIAL

CHOOSE $V(a, \theta, \varphi) = 0$

AT INFINITY, UNIFORM FIELD

$$V(\infty, \theta, \varphi) = -E_0 z = -E_0 r \cos\theta$$

GENERAL SOLUTION AT $r = a$

$$V = 0 = \sum_{m=0}^{\infty} A_m a^m P_m(\cos\theta)$$

$$+ \sum_{m=0}^{\infty} B_m a^{-(m+1)} P_m(\cos\theta)$$

MULTIPLY BOTH SIDES BY $P_n(\cos(\theta))$

AND INTEGRATE

$$\cancel{0 = A_m a^m \int_{-1}^{+1}} \qquad 0 = \sum_{m=0}^{\infty} \int_{-1}^{+1} A_m a^m P_m(\cos\theta) P_n(\cos\theta)\, d(\cos\theta)$$

$$0 = \int_{-1}^{+1} \qquad + \sum_{m=0}^{\infty} \int_{-1}^{+1} B_m a^{-(m+1)} P_m(\cos\theta) P_n(\cos\theta)\, d(\cos\theta)$$

ALL TERMS WITH $m \neq n$ VANISH

LEFT WITH

$$0 = A_n a^n \left(\frac{2}{2n+1}\right) + B_n a^{-(n+1)} \left(\frac{2}{2n+1}\right)$$

SOLVE $\qquad B_n = -A_n \, a^{2n+1}$

AS $r \to \infty$ ALL TERMS $r^{-(n+1)} \to 0$

LEFT WITH

$$-E_0 \, r \, P_1(\cos\theta) = \sum_{n=0}^{\infty} A_n r^n \, P_n(\cos\theta)$$

ONLY THE $n=1$ TERM ON RHS IS NON-ZERO

$\Rightarrow \qquad A_1 = -E_0$

ALL $A_n = 0 \qquad n \neq 1$

$\Rightarrow$ ALL $B_n = 0$ EXCEPT B_1

$$B_1 = -A_1 a^3 = E_0 a^3$$

$$V(r,\theta,\varphi) = -E_0 \left(1 - \frac{a^3}{r^3}\right) r \cos\theta$$

independent of φ

CALCULATE $\vec{E}$ FROM V

$$E_r = -\frac{\partial V}{\partial r} = E_0 \left(1 + \frac{2a^3}{r^2} \right) \cos\theta$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_0 \left(1 - \frac{a^3}{r^3} \right) \sin\theta$$

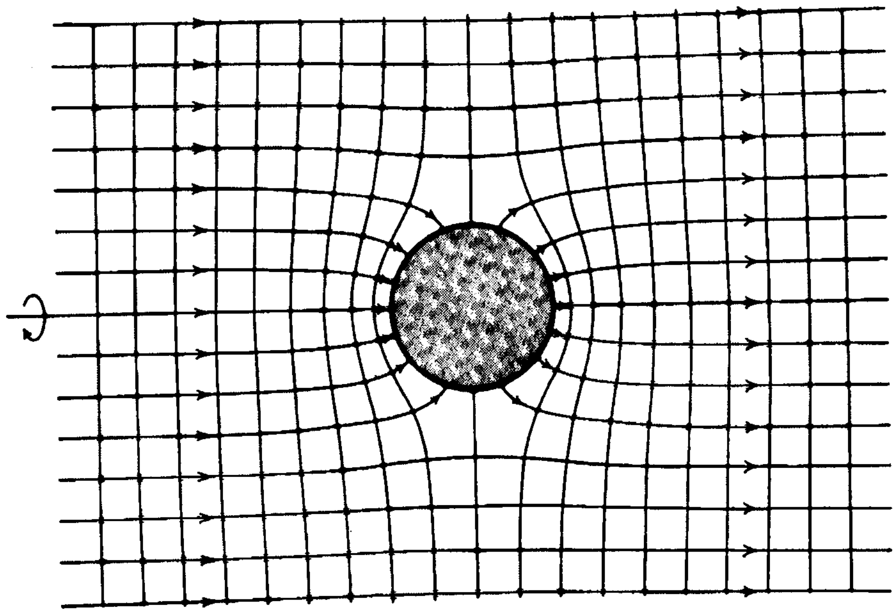


Figure 4-22. *Lines of force (indicated by arrows) and equipotentials for a conducting sphere in a uniform electrostatic field. The lines of force are normal at the surface of the sphere, and there is zero electric intensity inside. Observe that the field is hardly disturbed at distances larger than one radius from the surface of the sphere.*

insulated conducting sphere situated in a uniform electrostatic field $\mathbf{E}_0$, as in Figure 4-22.

At any point, either inside or outside the sphere, the electric field intensity is that due to the induced charges plus $\mathbf{E}_0$. We assume that the charges which produce $\mathbf{E}_0$ are so far away that they are unaffected by the presence of the sphere. The induced charges arrange themselves on the conducting sphere such that the total field is zero *inside*. Outside the sphere, the total field is of course not zero; we shall calculate it by solving Laplace's equation.

The field is best described in terms of spherical polar coordinates with the origin at the center of the sphere and the polar axis along $\mathbf{E}_0$. Our boundary conditions are then

$$V = 0 \qquad (r = a), \tag{4-141}$$

$$V = -E_0 z = -E_0 r \cos\theta \qquad (r = \infty). \tag{4-142}$$

At $r = a$, from Eqs. *4-139* and *4-141*,

$$0 = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta). \tag{4-143}$$

DIELECTRIC SPHERE

$V \rightarrow -E_0 r \cos\theta$ as $r \rightarrow \infty$

V is continuous across boundary at $r=a$

The normal component of D is continuous

OUTSIDE MUST HAVE ONLY $-E_0 r \cos\theta$

INSIDE: NO NEGATIVE POWERS OF r

OTHERWISE $V(r=0) \rightarrow \infty$

$$V_{OUTSIDE} = -E_0 r \cos\theta + \sum_{m=0}^{\infty} B_m r^{-(m+1)} P_m(\cos\theta)$$

$$V_{INSIDE} = \sum_{m=0}^{\infty} C_m r^m P_m(\cos\theta)$$

$$V_{OUT}(a,\theta,\varphi) = V_{in}(a,\theta,\varphi) \qquad \text{V CONTINOUS}$$

$$-\left.\frac{\partial V_o}{\partial r}\right|_{r=a} = -K_e \left.\frac{\partial V_i}{\partial r}\right|_{r=a} \qquad D_n \text{ CONTINUOUS}$$

1

$$-E_0 a P_1 + \frac{B_0}{a} + \frac{B_1 P_1}{a^2} + \frac{B_2 P_2}{a^3} + \cdots$$

$$= C_0 + C_1 a P_1 + C_2 a^2 P_2 + \cdots$$

AND

2 $$E_0 P_1 + \frac{B_0}{a^2} + \frac{2B_1 P_1}{a^3} + \frac{3B_2 P_2}{a^4} + \cdots$$

$$= -K_e C_1 P_1 - 2K_e C_2 a P_2 + \cdots$$

TRUE FOR ALL θ => COEFF'S ON LHS = COEFF'S ON RHS

FROM 1

$$\frac{B_0}{a} = C_0$$

$$-E_0 a + \frac{B_1}{a^2} = C_1 a$$

$$\frac{B^3}{a^3} = C_2 a^2$$

$$\vdots$$

FROM (2) $$\frac{B_2}{a^2} = 0$$

$$E_0 + \frac{2B_1}{a^3} = -K_e C_1$$

$$\frac{3B_2}{a^4} = -2K_e C_2 a$$

PUTTING (1) AND (2) TOGETHER

$$B_0 = C_0 = 0$$

$$B_1 = \left(\frac{K_e - 1}{K_e + 2}\right) E_0 a^3$$

$$C_1 = \frac{-3E_0}{K_e + 2}$$

$$B_n = C_n = 0 \qquad n > 1$$

SO

$$V_{OUT}(r,\theta,\varphi) = -\left[1 - \left(\frac{K_e - 1}{K_e + 2}\right)\frac{a^3}{r^3}\right] E_0 r \cos\theta$$

$$V_{IN}(r,\theta,\varphi) = -\left(\frac{3}{K_e + 2}\right) E_0 r \cos\theta = -\left(\frac{3}{K_e + 2}\right) E_0 z$$

$\vec{E}$ FIELD INSIDE IS UNIFORM AND IS ∥ TO $\hat{z}$.

$$E_0 + \frac{2B_1}{a^3} = -K_e C_1, \tag{4-184}$$

$$\frac{3B_2}{a^4} = -2K_e C_2 a. \tag{4-185}$$

These sets of equations lead to the following values for the coefficients:

$$B_0 = C_0 = 0, \tag{4-186}$$

$$B_1 = \left(\frac{K_e - 1}{K_e + 2}\right) E_0 a^3, \tag{4-187}$$

$$C_1 = -\frac{3E_0}{K_e + 2}, \tag{4-188}$$

$$B_n = C_n = 0 \qquad (n > 1). \tag{4-189}$$

Thus

$$V_0(r, \theta) = -\left[1 - \left(\frac{K_e - 1}{K_e + 2}\right)\frac{a^3}{r^3}\right] E_0 r \cos\theta, \tag{4-190}$$

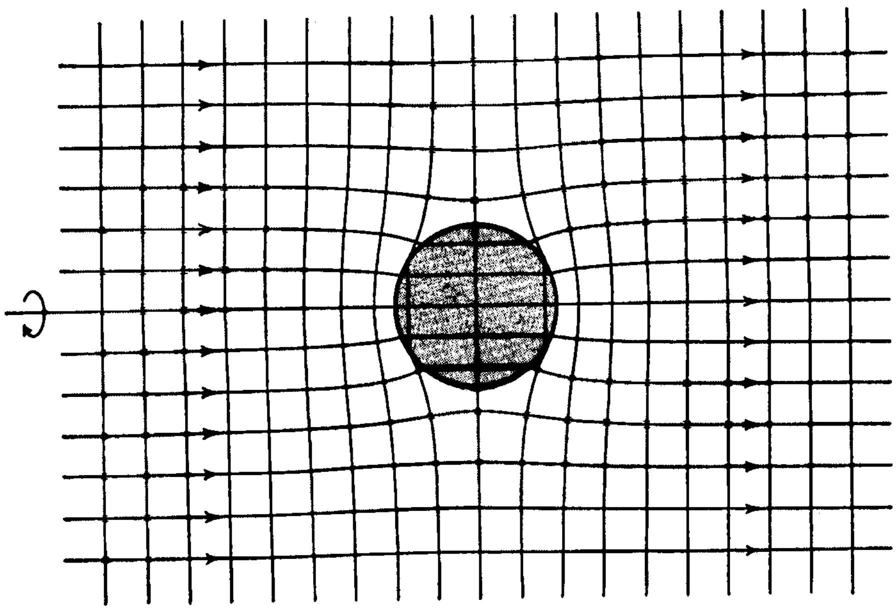


Figure 4-24. *The field near a dielectric sphere in a uniform electrostatic field. The lines of electric displacement (indicated by arrows) crowd into the sphere as shown, with the result that D is larger inside than outside. Since there is no free charge at the surface of the sphere, the lines of D neither originate nor terminate there, and they are continuous across the boundary. The equipotentials spread out inside, corresponding to a lower electric field intensity E. The electric field intensity E is discontinuous at the surface, and the density of lines of force is lower inside than outside. As in the conducting sphere, the field is hardly disturbed at distances larger than one radius from the surface. The field inside is uniform.*

POISSON'S EQUATION

$$\nabla^2 V = -\rho/\epsilon_0$$

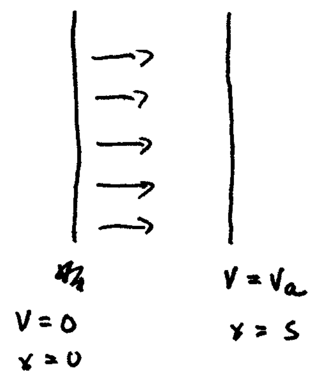
LAPLACE $\Rightarrow$ POTENTIAL IN REGIONS WHERE $\rho=0$

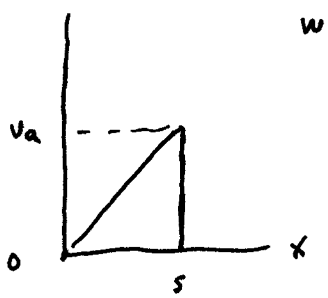
REGIONS WITH NO CHARGES

POISSON $\Rightarrow$ POTENTIAL IN REGIONS WHERE $\rho \neq 0$

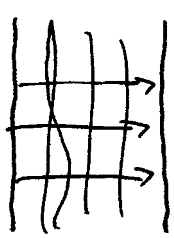
HOW CAN ρ NOT BE ZERO

CURRENT IS FLOWING





WHEN CHARGE FLOWING ~~IS NEGLIGIBLE~~
CAN BE NEGLECTED

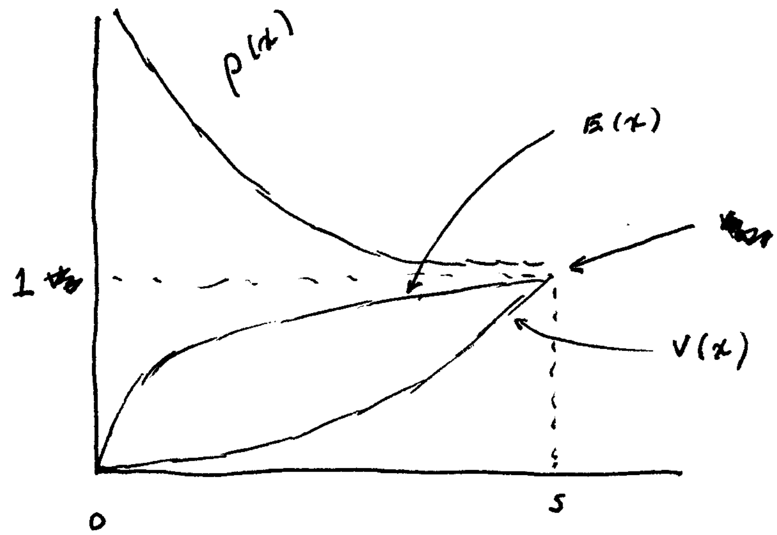


EQUALLY SPACED
EQUIPOTENTIALS
SURFACES

PERPENDICULAR
$\vec{E}$ FIELD LINES

DIODES (VACUUM AND SEMICONDUCTOR)

WHEN CHARGE IS FLOWING



NORMALIZED TO THEIR ANODE VALUES

$E \sim x^{1/3}$

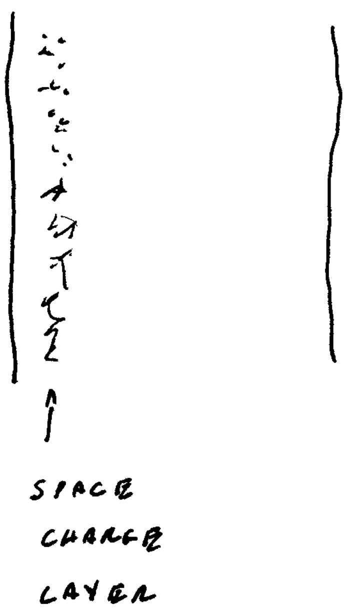
$I \sim V_0^{3/2}$

$V \sim x^{4/3}$

$$V(x) = V_0 \left(\frac{x}{s}\right)^{4/3}$$

$$E(x) = \frac{4}{3}\frac{V_0}{s}\left(\frac{x}{s}\right)^{1/3}$$

$$\rho(x) = \frac{4\epsilon_0}{9s^2}\left(\frac{x}{s}\right)^{-2/3}$$



CHILD-LANGMUIR LAW

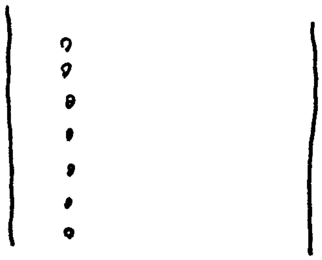
THREE-HALVES POWER LAW $\quad I \sim V^{3/2}$

DEBYE SCREENING

PLASMAS

SEMICONDUCTORS

ELECTROLYTIC SOLUTIONS



$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\left(e^{-r/r_0}\right)$$

Debye screening length