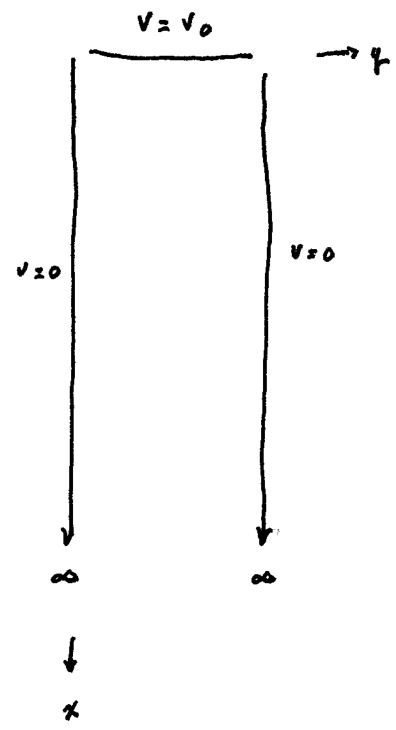
LECTURE 10 7-9-09

LAPLACE'S EQN IN 3d

$$\nabla^2 V = 0$$

FIRST, SUMMERIZE 1d FOURIER SERIES

$\boxed{\text{2d LAPLACE}}$



=

$$a_1 \sin(k_1 y) \; e^{-k_1 x}$$

$$a_3 \sin(k_3 y) \; e^{-k_3 x}$$

$$a_5 \sin(k_5 y) \; e^{-k_5 x}$$

$\vdots$

space (y) · space (x)

INTUITIVE METHOD

SAID: y OSCILLATING BY INSPECTION

x DECAYING

DON'T NEED TO DO THAT:

$\sin x$ $\cos x$ e^{-x} e^{+x} $\sinh x$

$\sin y$ $\cos y$ e^{-y} e^{+y} $\cosh y$

FORMAL METHOD

WRITE DOWN ALL OF THEM, THEN APPLY BC's

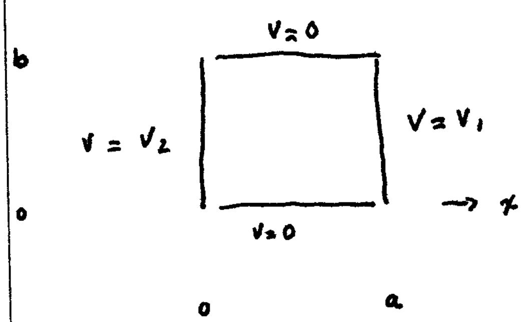
$$A \sin(kx)\sinh(ky)$$

$$+ B \cos(kx)\sinh(ky)$$

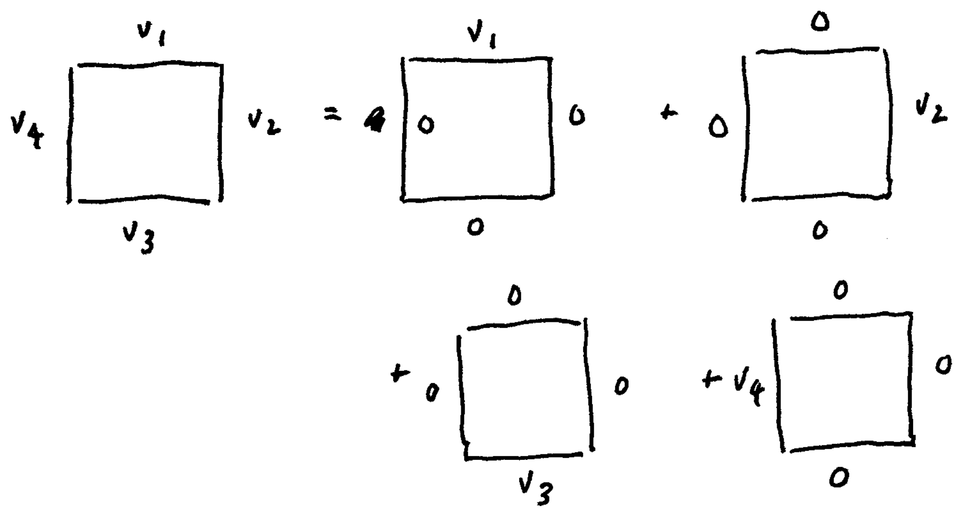
$$+ C \sin(ky)\sinh(kx)$$

$$+ D \cos(ky)\cosh(kx)$$

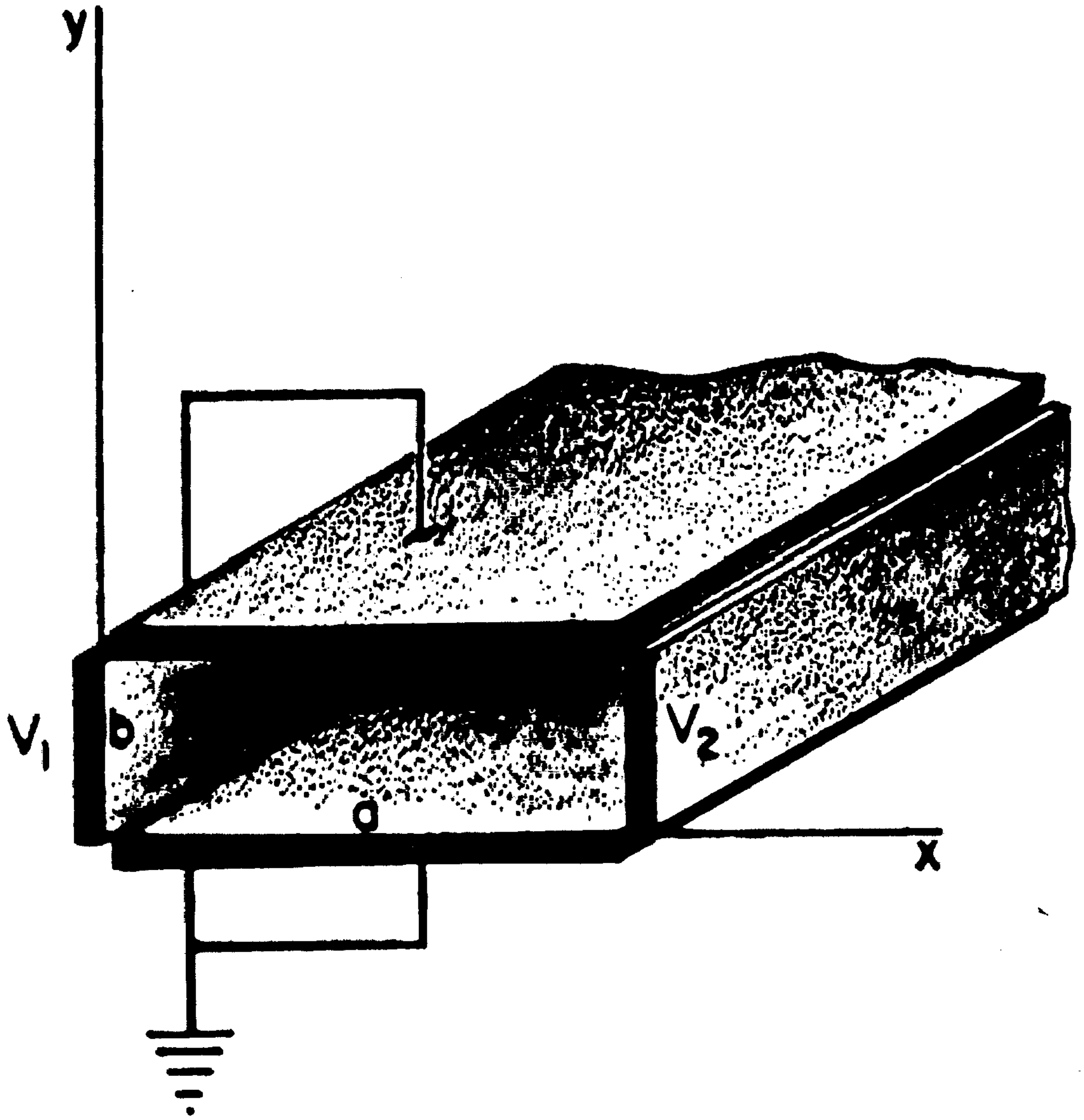
THEN APPLY BC'S

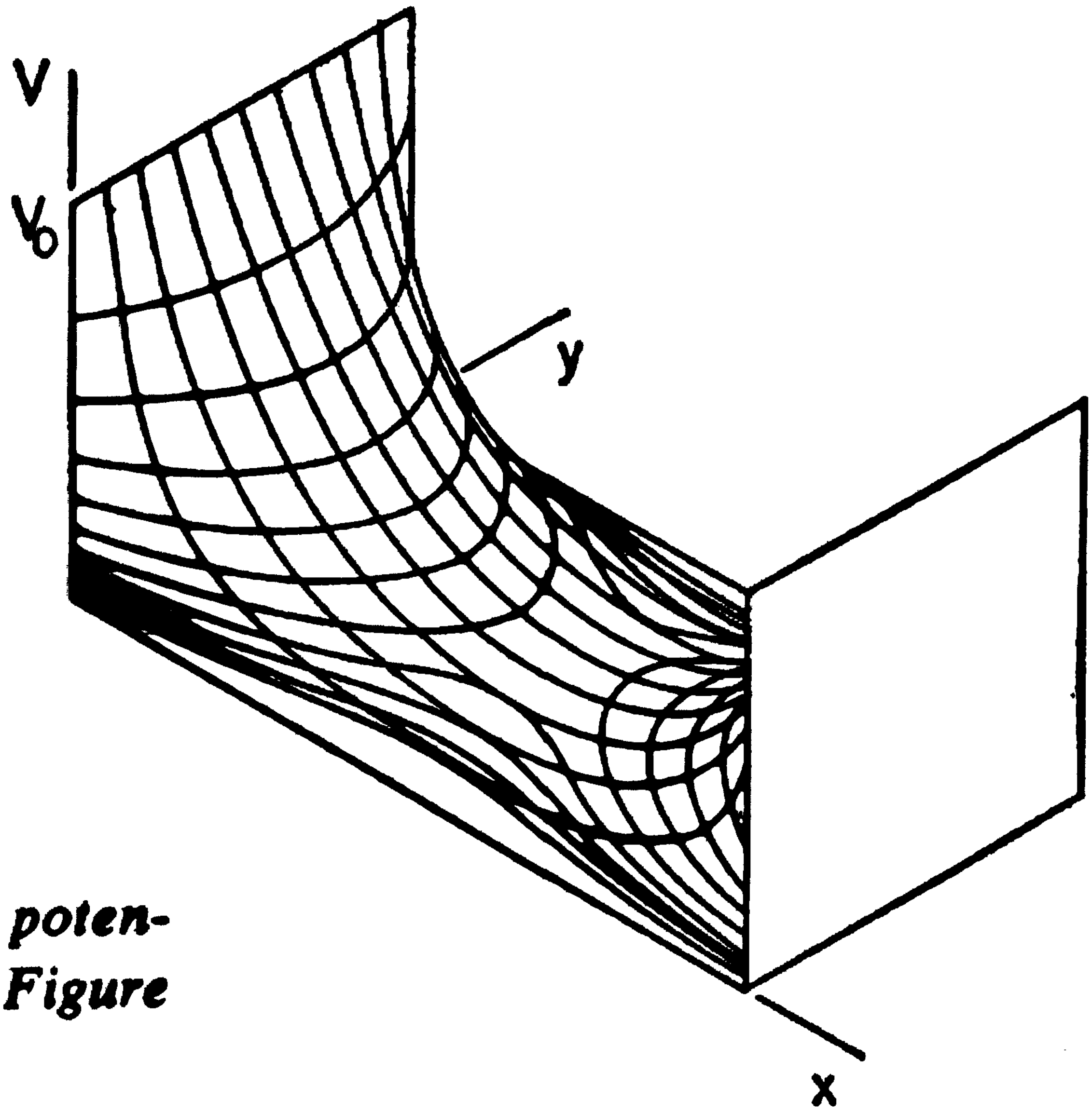


SUPERPOSITION



SOLVE 1 ⇒ GET THE OTHER 3 FOR FREE

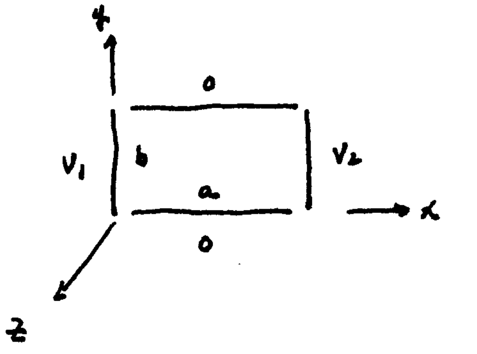

y
V_1
b
a
V_2
x



e poten-
f Figure

POST BOTH FROM CORSON

K $\boxed{\text{ON WEBSITE}}$



$$V(x,y,z) = \sum_n \left(A_n e^{-n\pi x/b} + B_n e^{+n\pi x/b} \right) \sin \frac{n\pi y}{a}$$

L BC $\quad A_n + B_n = \dfrac{4V_1}{n\pi} \quad n$ odd

$\qquad\qquad 0 \quad n$ even

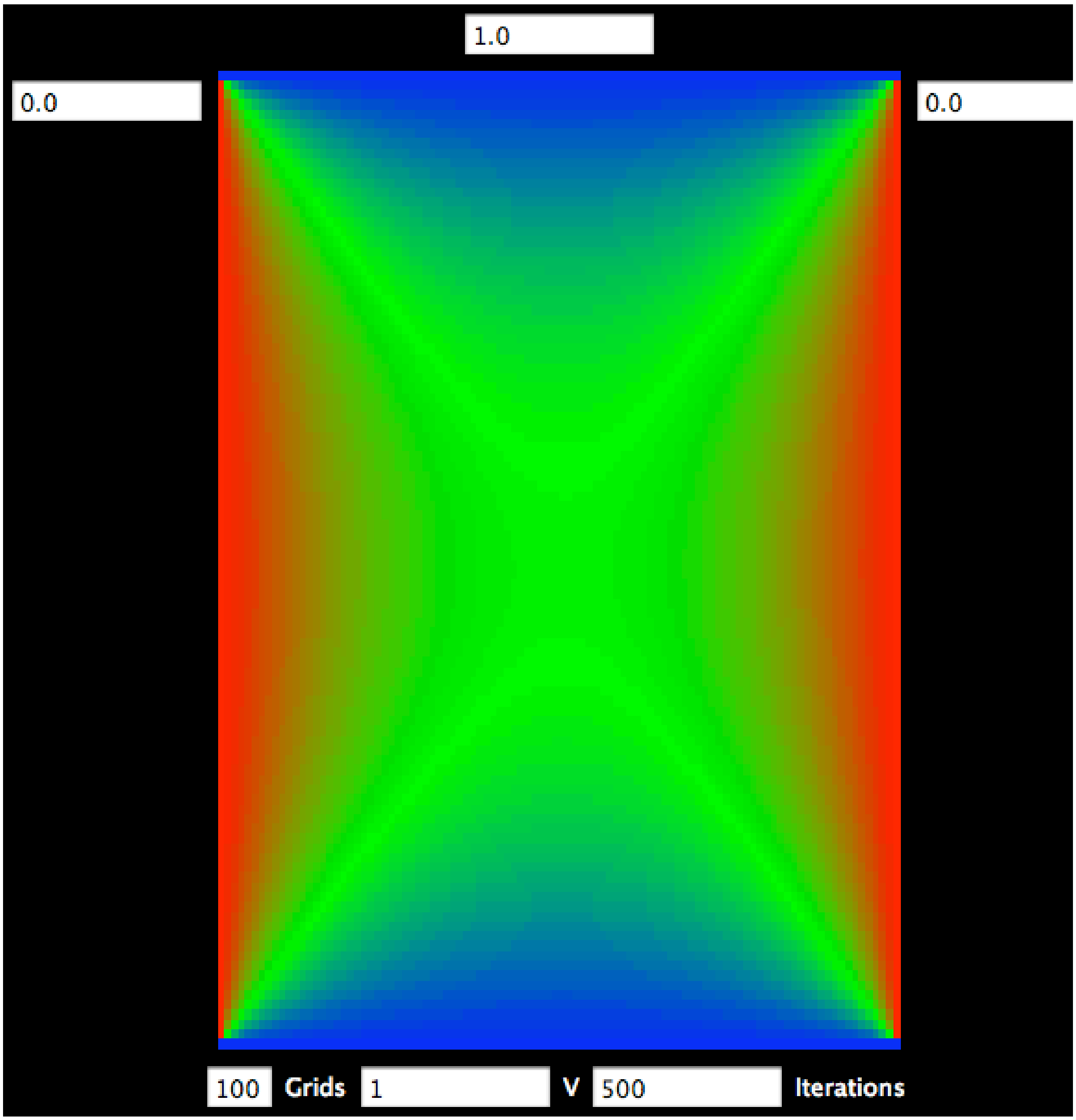
R BC $\quad A_n e^{-n\pi a/b} + B_n e^{+n\pi a/b} = \dfrac{4V_2}{n\pi} \quad n$ odd

$\qquad\qquad = 0 \quad n$ even

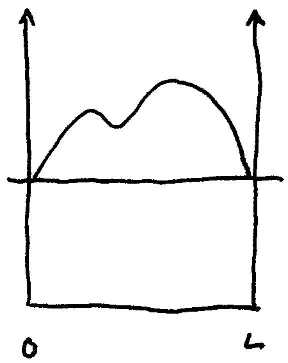
SOLUTION

$$A_n = \frac{4}{n\pi} \left(\frac{V_1 - V_2 e^{-n\pi a/b}}{1 - e^{-2n\pi a/b}} \right)$$

$$B_n = \frac{4 e^{-n\pi a/b}}{n\pi} \left(\frac{V_2 - V_1 e^{-n\pi a/b}}{1 - e^{-2n\pi a/b}} \right)$$


1.0
0.0
0.0
100
Grids
1
V
500
Iterations

1d SCHRODINGER



$$a_1 \sin(k_1 x) \quad e^{-iE_1 t/\hbar}$$

$+$

$$a_2 \sin(k_2 x) \quad e^{-iE_2 t/\hbar}$$

$+$

$$a_3 \sin(k_3 x) \quad e^{-iE_3 t/\hbar}$$

$+$

$\vdots$

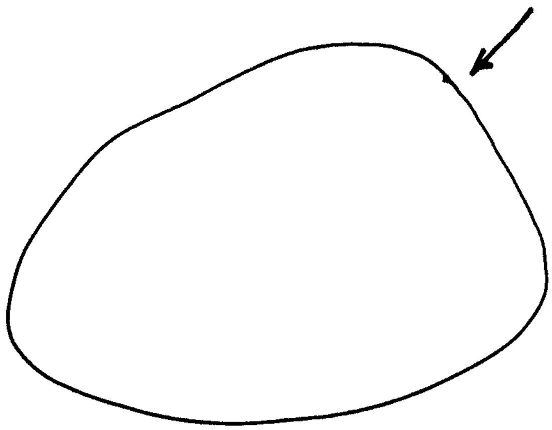
space (x) time (t)

BOUNDARY CONDITIONS

DIRICHLET BCs

~~FIRST~~ TYPE I BCs

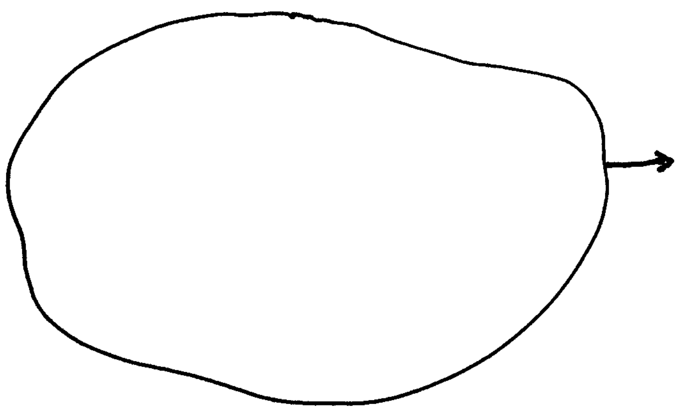
V on boundary



FIXED VALUE

NEUMANN BCs

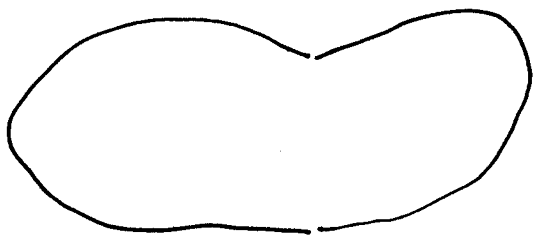
~~SECOND~~ TYPE II BCS



$\frac{\partial V}{\partial n}$ on boundary

FIXED FLOW

MIXED BCS



TYPE 1

TYPE 2

CAUCHY BCs

KNOW BOTH	VALUE	WAVE FCN
	NORMAL DERIVATIVE	SLOPE OF WAVE FCN

in 3d

THREE STANDARD COORDINATE SYSTEMS

CARTESIAN x, y, z

SPHERICAL r, θ, φ

CYLINDRICAL ρ, θ, z

∇^2 LOOKS DIFFERENT!

SOLUTIONS ARE DIFFERENT

OF COURSE, YOU CAN USE ANY COORDINATE SYSTEM

"IF THE BOUNDARY CONDITIONS ARE

NOT SEPARABLE, MOST LIKELY

WE ARE HOSED."

$\longrightarrow$ NEXT PAGE

http://quantumrelativity.calsci.com/Physics/EandM7.html

If the boundary conditions are not separable, most likely we're hosed

Generally speaking, if the boundary conditions are separable, there's a good chance the solution is separable. If the boundary conditions are not separable, most likely we're hosed.

This is Bessel's equation. The solutions are Bessel functions, Neumann functions, and Hankel functions, and we've officially entered Graduate Student Hell.

http://www.urbandictionary.com/define.php?term=hosed

IN 3d $\nabla^2 V = 0$

SEPARATES IN 11 + 2 = 13 COORD SYSTEMS

$V(x,y,z) = X(x)\, Y(y)\, Z(z)$ CARTESIAN

$V(r,\theta,\varphi) = R(r)\, \Theta(\theta)\, \Phi(\varphi)$ SPHERICAL

$V(r,\theta,z) = R(r)\, \Theta(\theta)\, Z(z)$ CYLINDRICAL

ONE EQN FOR EACH COORD

$$\frac{d^2 X}{dx^2} = c_1 X$$

$$\frac{d^2 Y}{dy^2} = c_2 Y$$

$$\frac{d^2 Z}{dz^2} = c_3 Z$$

$$c_1 + c_2 + c_3 = 0$$

Coordinate System	Variables	Solution Functions
Cartesian	$X(x) Y(y) Z(z)$	exponential functions, circular functions, hyperbolic functions
circular cylindrical	$R(r) \Theta(\theta) Z(z)$	Bessel functions, exponential functions, circular functions
conical		ellipsoidal harmonics, power
ellipsoidal	$\Lambda(\lambda) M(\mu) N(\nu)$	ellipsoidal harmonics
elliptic cylindrical	$U(u) V(v) Z(z)$	Mathieu function, circular functions
oblate spheroidal	$\Lambda(\lambda) M(\mu) N(\nu)$	Legendre polynomial, circular functions
parabolic		Bessel functions, circular functions
parabolic cylindrical		parabolic cylinder functions, Bessel functions, circular functions
paraboloidal	$U(u) V(v) \Theta(\theta)$	circular functions
prolate spheroidal	$\Lambda(\lambda) M(\mu) N(\nu)$	Legendre polynomial, circular functions
spherical	$R(r) \Theta(\theta) \Phi(\phi)$	Legendre polynomial, power, circular functions

Laplace's equation can be solved by separation of variables in all 11 coordinate systems that the Helmholtz differential equation can. The form these solutions take is summarized in the table above. In addition to these 11 coordinate systems, separation can be achieved in two additional coordinate systems by introducing a multiplicative factor. In these coordinate systems, the separated form is

Cartesian Coordinates

RECTANGULAR COORDINATES

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Assume we may write

$$\Phi(x,y,z)=X(x)Y(y)Z(z)$$

$$YZ\frac{d^2X}{dx^2}+XZ\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}=0$$

Note that the derivatives are no longer partial.

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

The first term depends on x only, the second on y only and the third on z only. The equation can only be valid if each of the terms is a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2}=\alpha'^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=\beta'^2$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=\gamma'^2$$

$$\alpha'^2+\beta'^2+\gamma'^2=0$$

Since we are considering the electrostatic potential it is real valued. This means that all these squares are real valued, but the last relation shows that the constants themselves cannot all be real valued, neither can they all be imaginary.

We can only have the following cases

a) two real, one imaginary
b) one real, two imaginary
c) one real, one imaginary, one zero
d) three zero

An imaginary separation constant leads to an oscillatory solution while a real valued leads to an exponential.

Let us arbitrarily let α' and β' be imaginary:

$$\alpha'^2 \equiv -\alpha^2$$
$$\beta'^2 \equiv -\beta^2$$
$$\gamma'^2 \equiv \gamma^2$$

α, β and γ are all real valued.

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0$$
$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0$$
$$\frac{d^2Z}{dz^2} - \gamma^2 Z = 0$$
$$\gamma^2 = \alpha^2 + \beta^2 \ \ ; \ \ \gamma = \sqrt{\alpha^2 + \beta^2}$$

$$X(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$$
$$Y(y) = Ce^{i\beta y} + De^{-i\beta y}$$
$$Z(z) = Ee^{\gamma z} + Fe^{-\gamma z}$$

The complete solution is

$$\Phi(x,y,z) = X(x)Y(y)Z(z)$$

$$\sum_{r,s=1}^{\infty}\left(A_r e^{i\alpha_r x} + B_r e^{-i\alpha_r x}\right)\left(C_s e^{i\beta_s y} + D_s e^{-i\beta_s y}\right)$$

$$\cdot\left(E_{rs} e^{\gamma_{rs} z} + F_{rs} e^{-\gamma_{rs} z}\right)$$

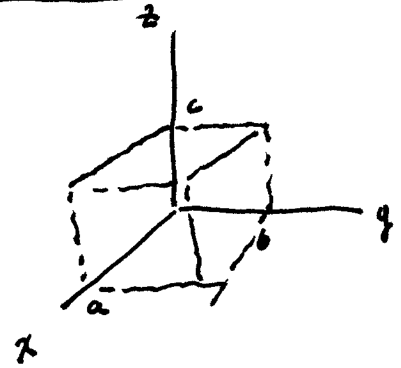
Short hand notation:

$$\boxed{\Phi(x,y,z) \sim e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z}}$$

All the constants will be determined from the boundary conditions of the problem.

5 FACES GROUNDED

1 FACE ARBITRARY



5 $V = 0$

1 $V(x,y,c) = V(x,y)$

$$X(x) = \sin \alpha x$$

$$Y(y) = \sin \beta y$$

$$Z(z) = \sinh \sqrt{\alpha^2 + \beta^2}\, z$$

$$\alpha_n = \frac{n\pi}{a}$$

$$\beta_m = \frac{m\pi}{b}$$

$$\gamma_{nm} = \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$V_{nm} = \sin(\alpha_n x)\,\sin(\beta_m y)\,\sinh(\gamma_{nm} z)$$

$$V(x,y,c) = \sum_{n,m} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$$

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\; V(x,y) \sin(\alpha_n x) \sin(\beta_m y)$$

2d FOURIER