Diffraction and the Fourier Transform

Light bends!

Diffraction assumptions

Solution to Maxwell's
  Equations

The near field

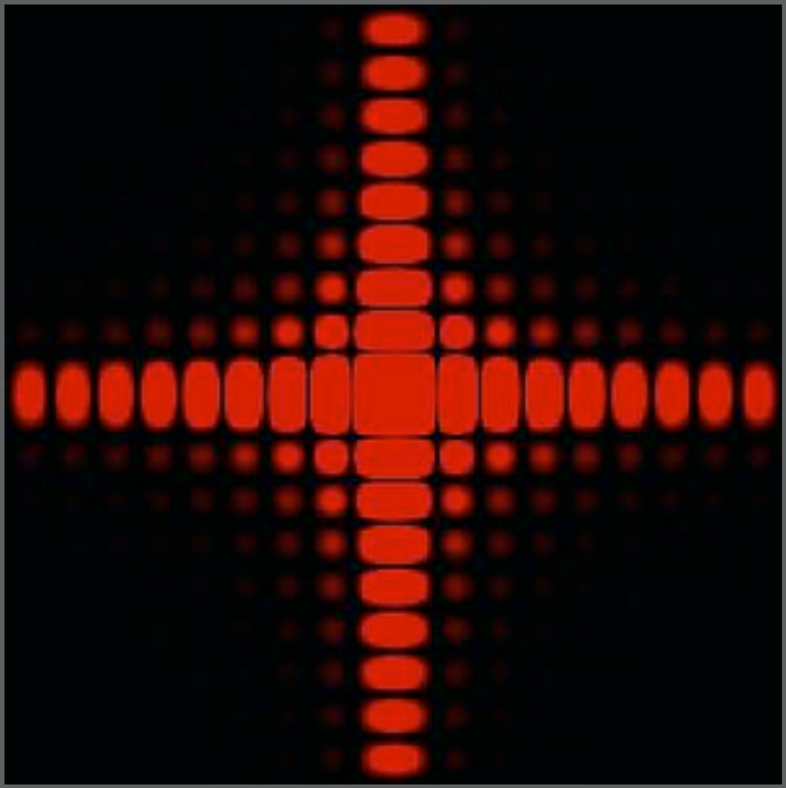
  Fresnel Diffraction
  Some examples

The far field

  Fraunhofer Diffraction
  Some examples

Young's two-slit experiment

Prof. Rick Trebino
Georgia Tech



www.physics.gatech.edu/frog/lectures

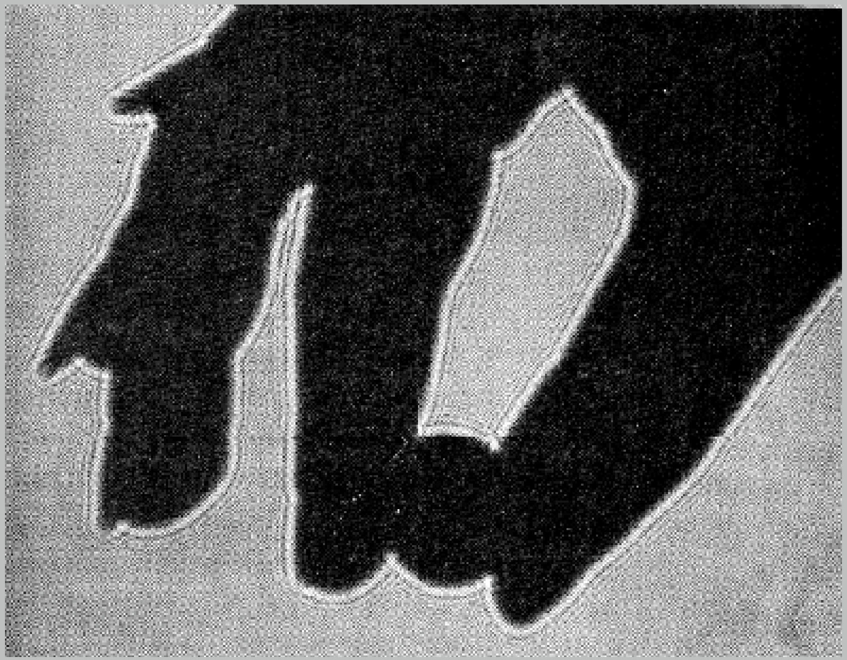
Diffraction

Light does not always travel in a straight line.

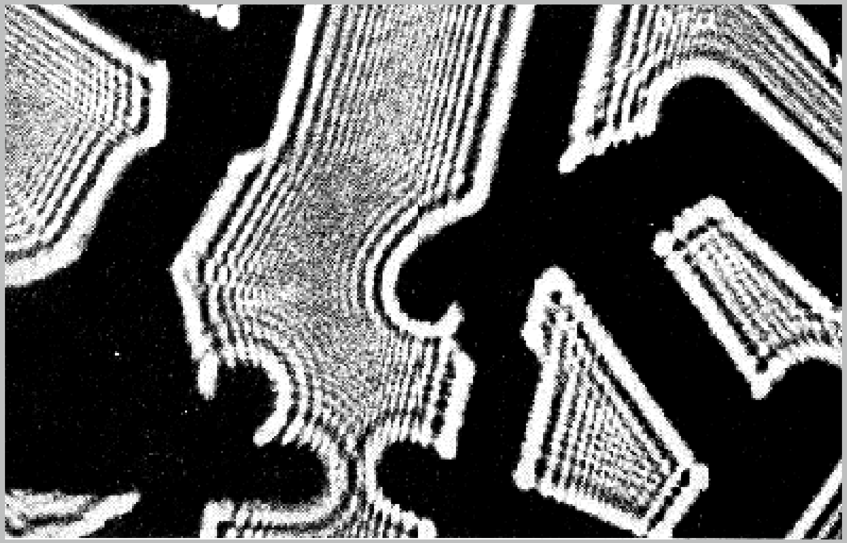
It tends to bend around objects. This tendency is called **diffraction**.

Any wave will do this, including matter waves and acoustic waves.

Shadow of a hand illuminated by a Helium-Neon laser

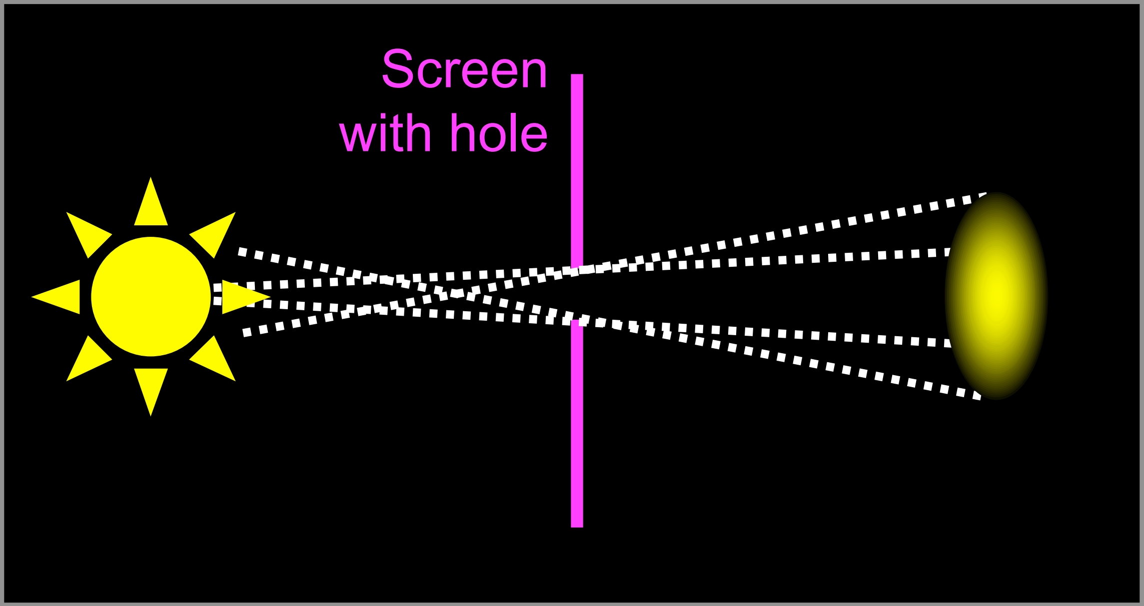


Shadow of a zinc oxide crystal illuminated by a electrons



Why it's hard to see diffraction

Diffraction tends to cause ripples at edges. But a point source is required to see this effect. A large source masks them.



Rays from a point source yield a perfect shadow of the hole. Rays from other regions blur the shadow.

Example: a large source (like the sun) casts blurry shadows, masking the diffraction ripples.

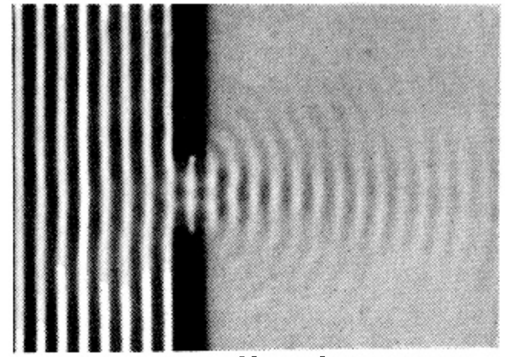
Diffraction of ocean water waves

Ocean waves passing through slits in Tel Aviv, Israel

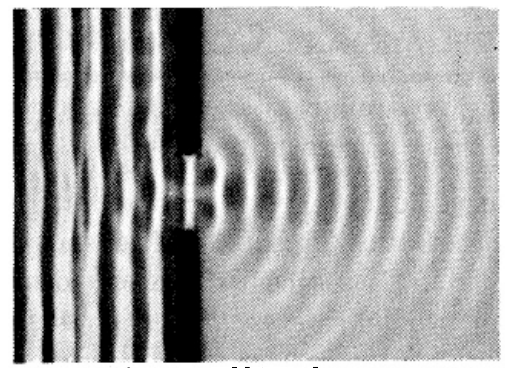
Diffraction occurs for all waves, whatever the phenomenon.

Diffraction of a wave by a slit

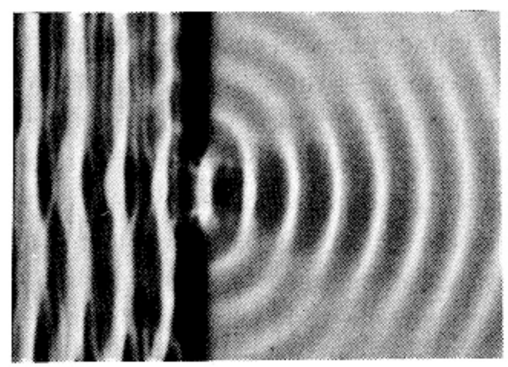
Whether waves in water or electromagnetic radiation in air, passage through a slit yields a diffraction pattern that will appear more dramatic as the size of the slit approaches the wavelength of the wave.



λ << slit size

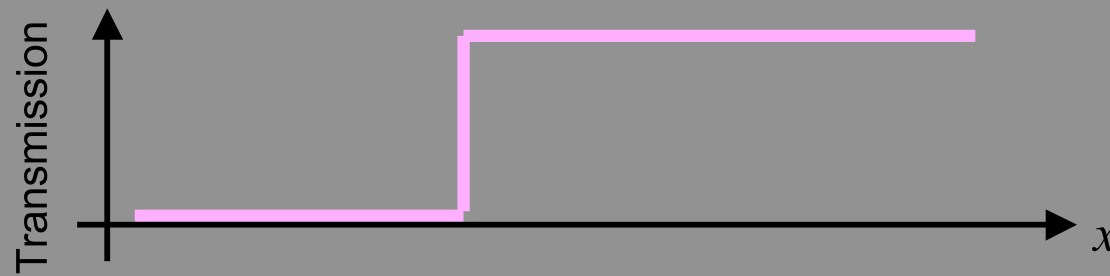


λ < slit size



$\lambda \approx$ slit size

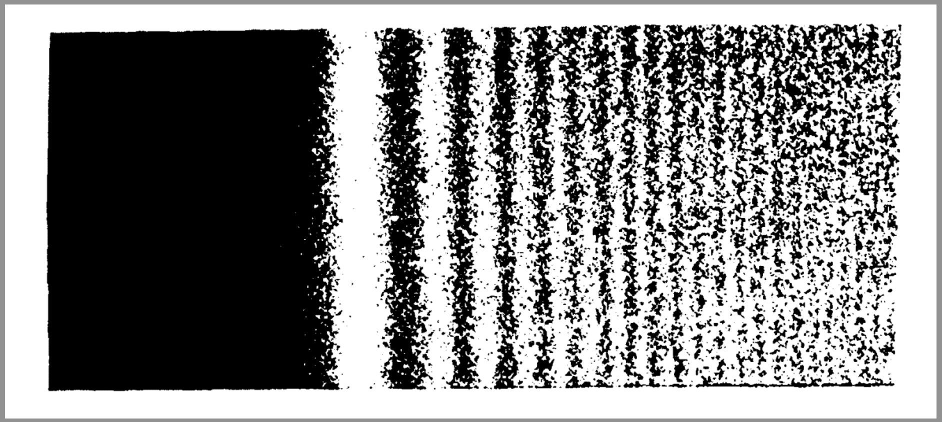
Diffraction by an Edge



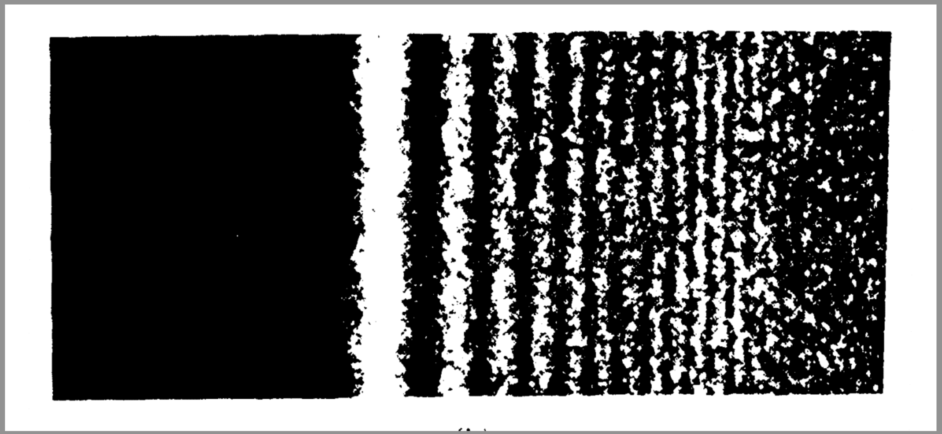
Even without a small slit, diffraction can be strong.

Simple propagation past an edge yields an unintuitive irradiance pattern.

Light passing by edge



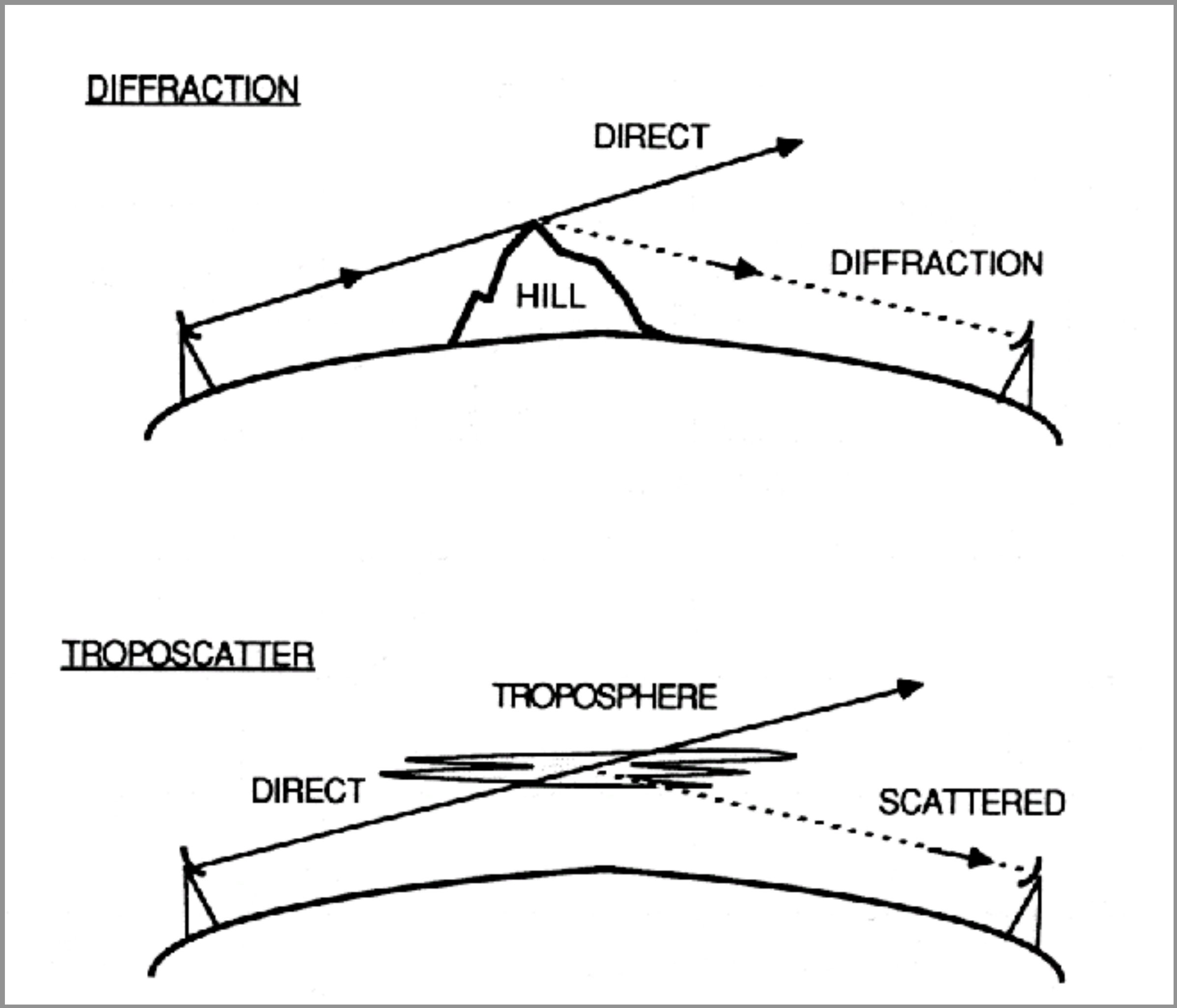
Electrons passing by an edge (Mg0 crystal)



Radio waves diffract around mountains.

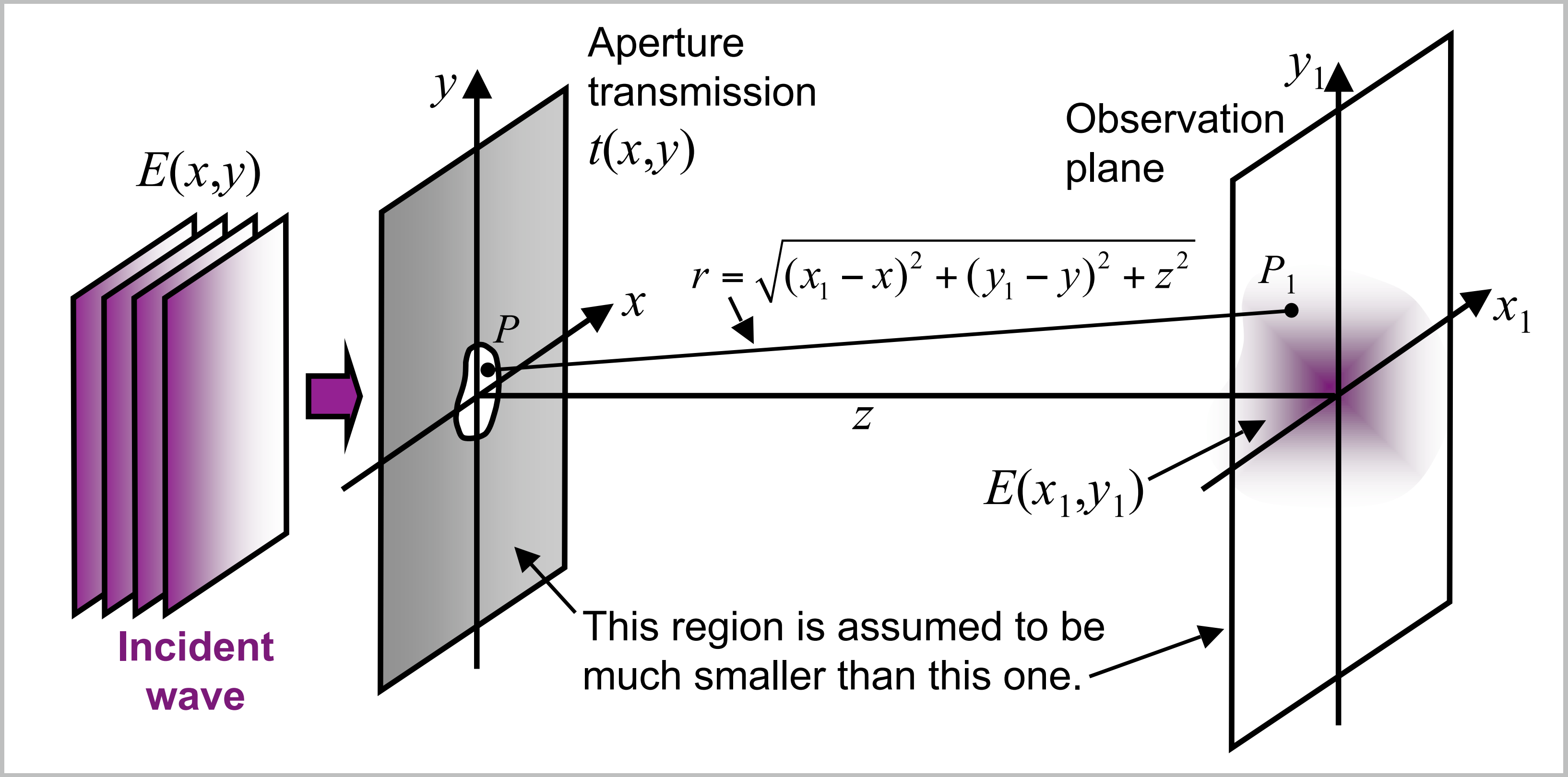
When the wavelength is a km long, a mountain peak is a very sharp edge!

Another effect that occurs is scattering, so diffraction's role is not obvious.



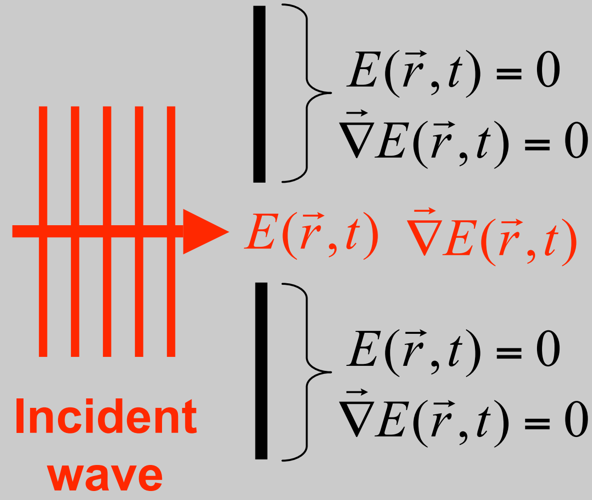
Diffraction Geometry

We wish to find the light electric field after a screen with a hole in it.
This is a very general problem with far-reaching applications.



What is $E(x_1,y_1)$ at a distance z from the plane of the aperture?

Diffraction Assumptions



The best assumptions were determined by Kirchhoff:

1) Maxwell's equations

2) Inside the aperture, the field and its spatial derivative are the same as if the screen were not present.

3) Outside the aperture (in the shadow of the screen), the field and its spatial derivative are zero.

While these assumptions give the best results, they actually over-determine the problem and can be shown to yield zero field everywhere! Nevertheless, we still use them.

Diffraction Solution

The field in the observation plane, $E(x_1,y_1)$, at a distance z from the aperture plane is given by a **convolution**:

$$E(x_1,y_1) = \iint h(x_1 - x, y_1 - y)\, t\left(x,y\right) E\left(x,y\right) dx\, dy$$

where:

$$h(x_1 - x, y_1 - y) = \frac{1}{i\lambda}\frac{\exp(ikr)}{r}$$

← Spherical wave!

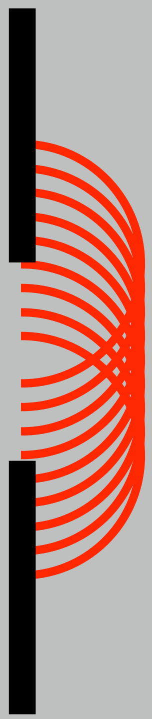
and:

$$r = \sqrt{z^2 + \left(x - x_1\right)^2 + \left(y - y_1\right)^2}$$

A very complicated result!

Huygens' Principle

Huygens' Principle says that every point along a wave-front emits a spherical wave that interferes with all others.



Christiaan Huygens
1629 – 1695

Our solution for diffraction illustrates this idea, and it's more rigorous.

Fresnel Diffraction: Approximations

In the denominator, we can approximate r by z. But we can't approximate r in the exp by z because it gets multiplied by k, which is big, so relatively small changes in r can make a big difference! But we can write:

$$r = \sqrt{z^2 + \left(x - x_1\right)^2 + \left(y - y_1\right)^2} = z\sqrt{1 + \left(\frac{x - x_1}{z}\right)^2 + \left(\frac{y - y_1}{z}\right)^2}$$

And if $\varepsilon << 1,\ \sqrt{1+\varepsilon} \approx 1 + \varepsilon / 2$

$$r \approx z\left[1 + \frac{1}{2}\left(\frac{x - x_1}{z}\right)^2 + \frac{1}{2}\left(\frac{y - y_1}{z}\right)^2\right] = z + \frac{\left(x - x_1\right)^2}{2z} + \frac{\left(y - y_1\right)^2}{2z}$$

This yields: $E(x_1, y_1) =$

$$\iint \frac{1}{i\lambda z} \exp\left\{ik\left[z + \frac{\left(x - x_1\right)^2}{2z} + \frac{\left(y - y_1\right)^2}{2z}\right]\right\} t\left(x, y\right) E\left(x, y\right) dx\, dy$$

Fresnel Diffraction: Approximations

Multiplying out the squares: $E\left(x_1, y_1\right) =$

$$\iint \frac{1}{i\lambda z} \exp\left\{ik\left[z + \frac{(x^2 - 2xx_1 + x_1^2)}{2z} + \frac{(y^2 - 2yy_1 + y_1^2)}{2z}\right]\right\} t\left(x, y\right) E\left(x, y\right) dx\, dy$$

Factoring out the quantities independent of x and y: $E\left(x_1, y_1\right) =$

$$\frac{\exp(ikz)}{i\lambda z} \exp\left[ik \frac{x_1^2 + y_1^2}{2z}\right] \iint \exp\left\{ik\left[\frac{(-2xx_1 - 2yy_1)}{2z} + \frac{(x^2 + y^2)}{2z}\right]\right\} t\left(x, y\right) E\left(x, y\right) dx\, dy$$

This is the **Fresnel integral**.

It yields the light wave field at the distance z from the screen.

Diffraction Conventions

We'll typically assume that a plane wave is incident on the aperture.

$$E(x, y) = constant$$

It still has an $\exp[i(\omega t - kz)]$, but it's constant with respect to x and y.

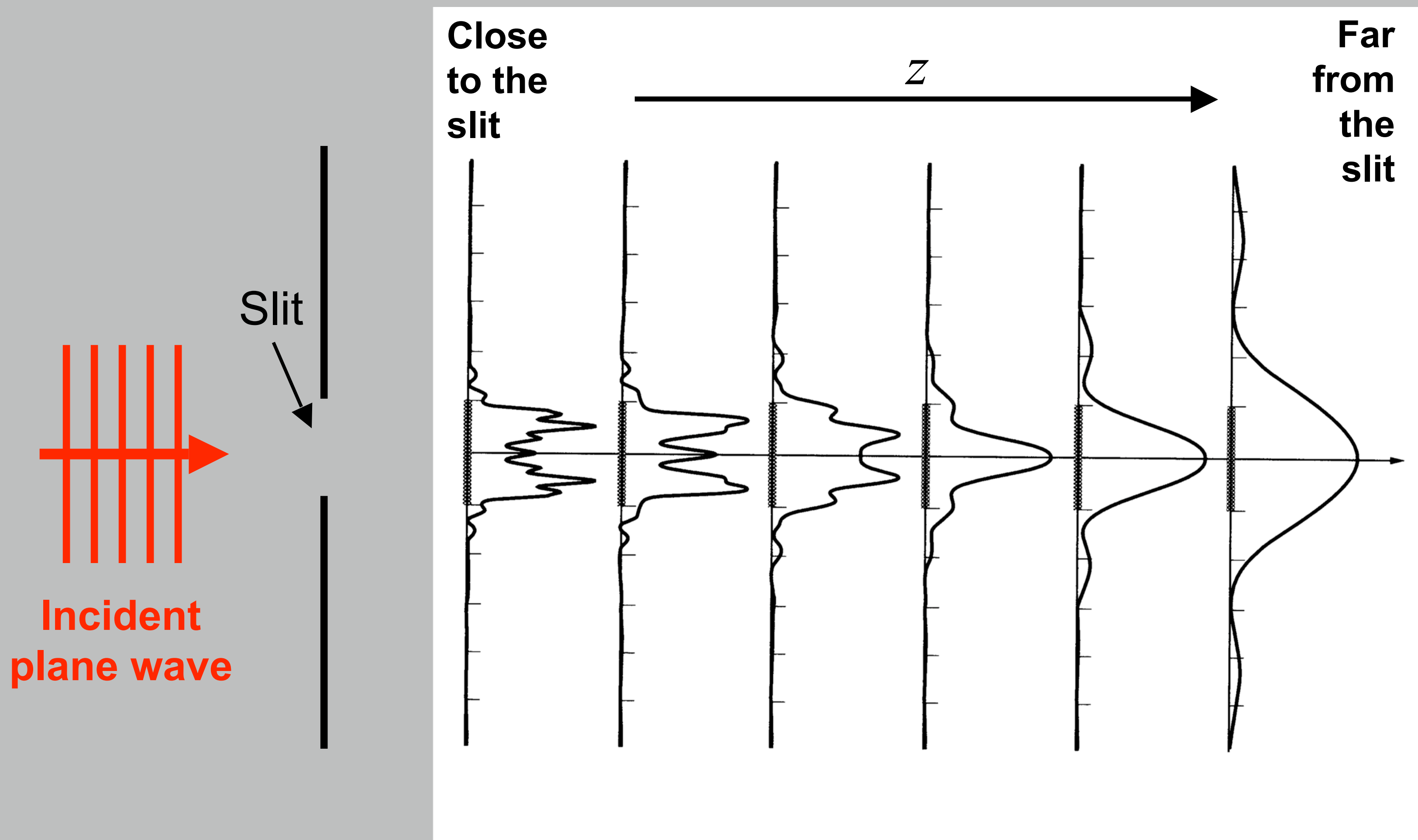
$$E(x_1, y_1) \propto \underbrace{\frac{\exp(ikz)}{i\lambda z} \exp\left[ik \frac{x_1^2 + y_1^2}{2z}\right]} \iint \exp\left\{ik\left[\frac{(-2xx_1 - 2yy_1)}{2z} + \frac{(x^2 + y^2)}{2z}\right]\right\} t(x, y)\, dx\, dy$$

And we'll usually ignore the various factors in front:

$$E(x_1, y_1) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{ik\left[\frac{(-2xx_1 - 2yy_1)}{2z} + \frac{(x^2 + y^2)}{2z}\right]\right\} t(x, y)\, dx\, dy$$

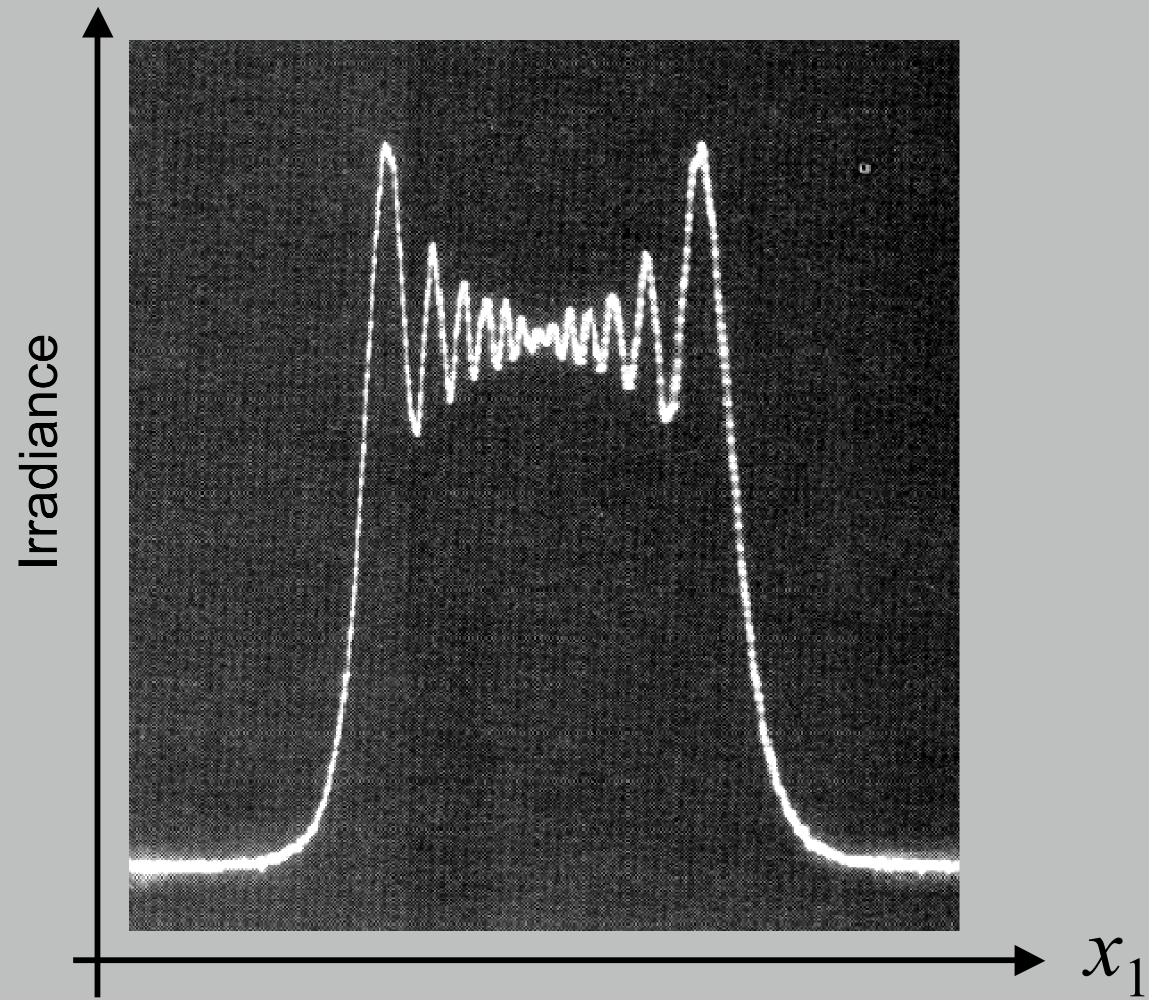
Fresnel diffraction: example

Fresnel diffraction from a single slit:



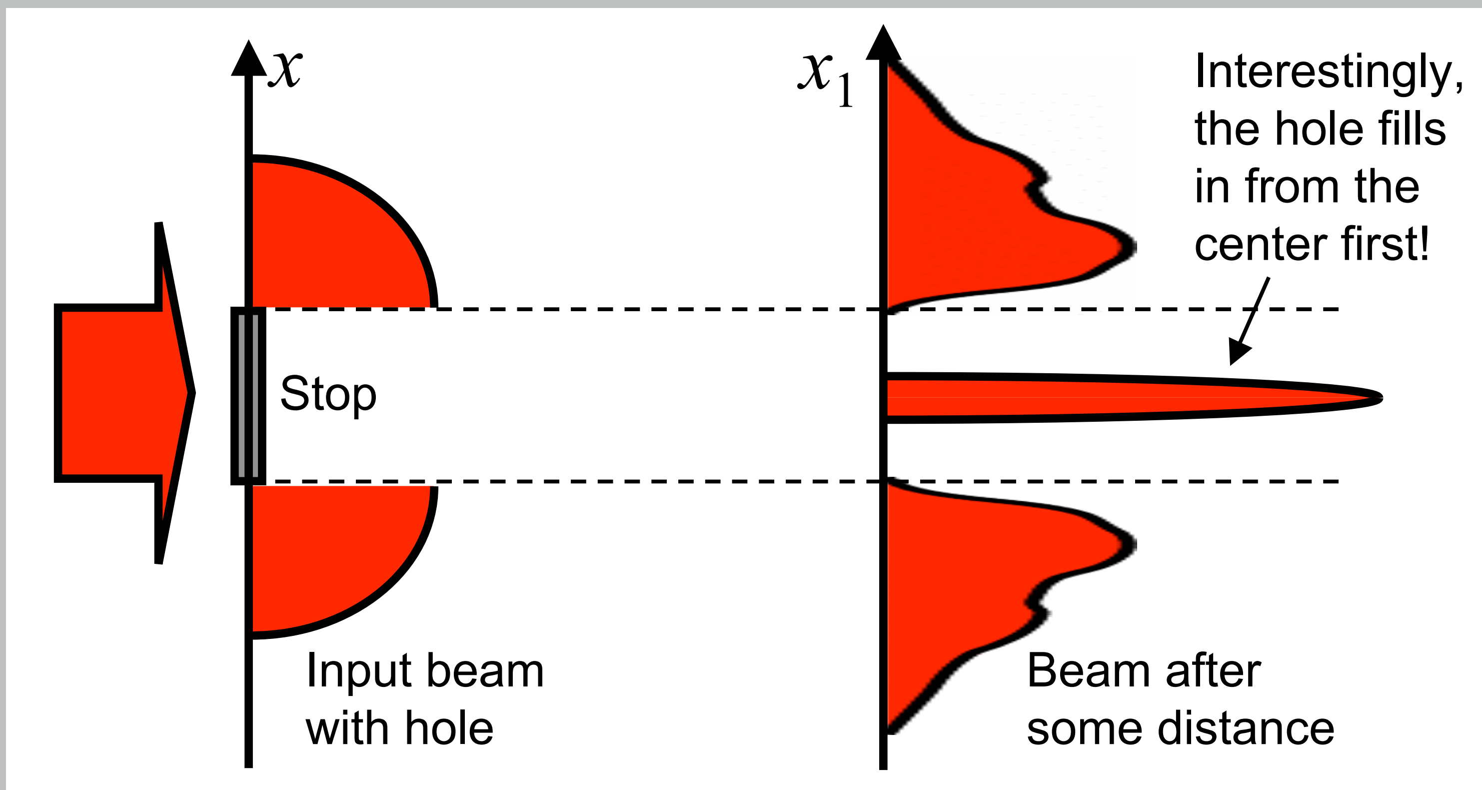
Fresnel Diffraction from a Slit

This irradiance vs. position just after a slit illuminated by a laser.



The Spot of Arago

If a beam encounters a stop, it develops a hole, which fills in as it propagates and diffracts:

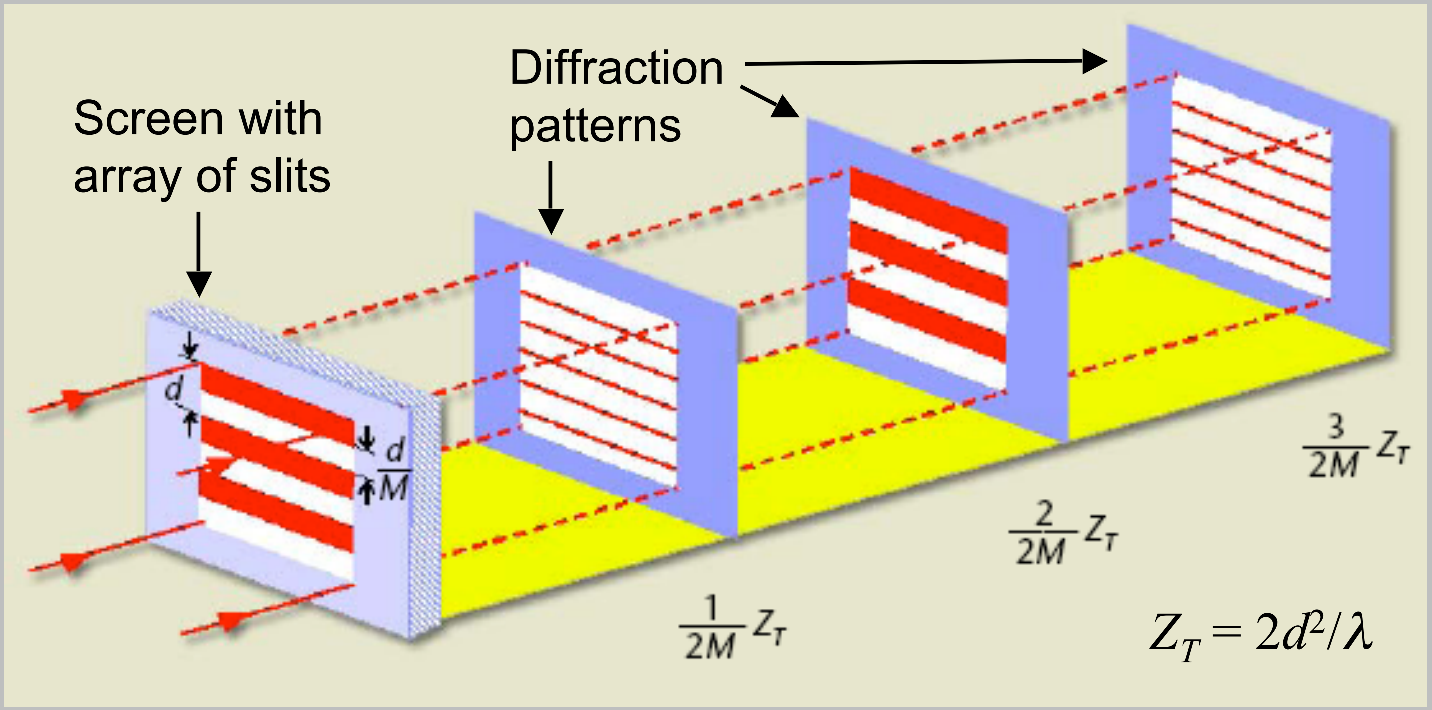


This irradiance can be quite high and can do some damage!

Fresnel diffraction from an array of slits: The Talbot Effect

One of the few Fresnel diffraction problems that can be solved analytically is an array of slits.

The beam pattern alternates between two different fringe patterns.

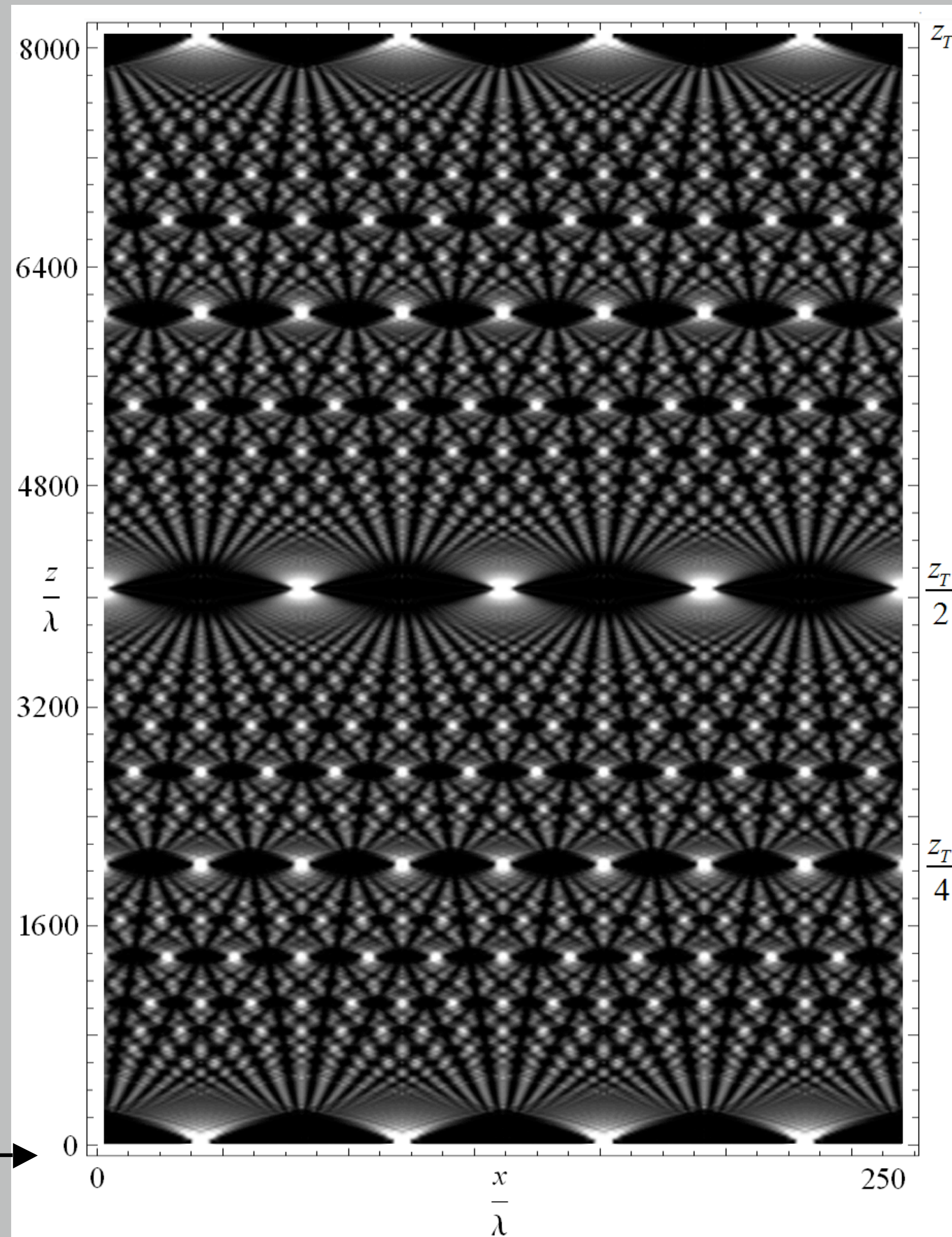


The Talbot Carpet

What goes on in between the solvable planes?

The beam propagates in this direction.

The slits are here.

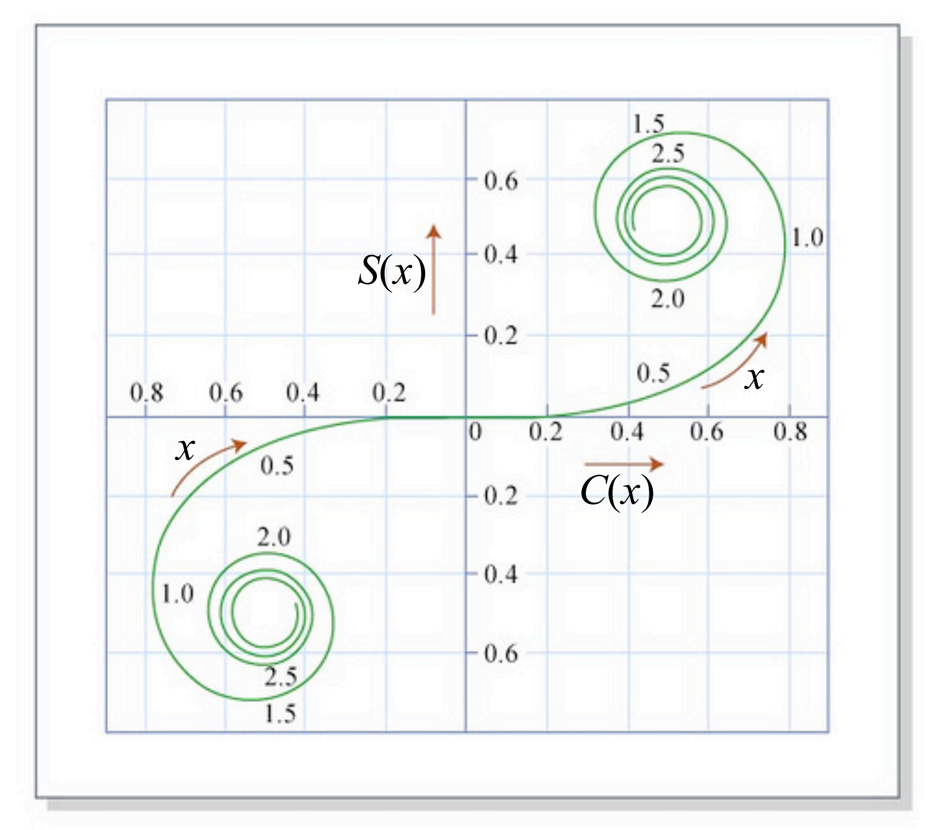


Diffraction Approximated

These integrals come up:

$$C(x) = \int_0^x \cos(t^2)\,dt$$

$$S(x) = \int_0^x \sin(t^2)\,dt$$



Such effects can be modeled by measuring the distance on a **Cornu Spiral.**

But most useful diffraction effects do not occur in the Fresnel diffraction regime because it's too complex.

For a cool Java applet that computes Fresnel diffraction patterns, try http://falstad.com/diffraction/

Fraunhofer Diffraction: The Far Field

Recall the Fresnel diffraction result: $E\left(x_1, y_1\right) =$

$$\frac{\exp(ikz)}{i\lambda z} \exp\left[ik\frac{x_1^2 + y_1^2}{2z}\right] \iint \exp\left\{ik\left[\frac{(-2xx_1 - 2yy_1)}{2z} + \frac{(x^2 + y^2)}{2z}\right]\right\} t\left(x, y\right) E\left(x, y\right) dx\, dy$$

Let D be the size of the aperture: $D^2 \geq x^2 + y^2$.

When $kD^2/2z << 1$, the quadratic terms << 1, so we can neglect them:

$$E\left(x_1, y_1\right) = \frac{\exp(ikz)}{i\lambda z} \exp\left[ik\frac{x_1^2 + y_1^2}{2z}\right] \iint \exp\left\{-\frac{ik}{z}\left(xx_1 + yy_1\right)\right\} t\left(x, y\right) E\left(x, y\right) dx\, dy$$

This condition means going a distance away: $z >> kD^2/2 = \pi D^2/\lambda$

If $D = 1$ mm and λ = 1 micron, then $z >> 3$ m.

Fraunhofer Diffraction Conventions

Neglect the phase factors, and we'll explicitly write the aperture transmission function, $t(x,y)$, in the integral:

$$E(x_1, y_1) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{ik}{z}(xx_1 + yy_1)\right\} t(x,y)\, E(x,y)\, dx\, dy$$

$E(x,y)$ = constant if a plane wave

This is just a Fourier Transform!

Interestingly, it's a Fourier Transform from position, x, to another position variable, x_1 (in another plane). Usually, the Fourier "conjugate variables" have reciprocal units (e.g., t & ω, or x & k). The conjugate variables here are really x and $k_x = kx_1/z$, which have reciprocal units.

So the far-field light field is the Fourier Transform of the transmitted field!

The Fraunhofer Diffraction formula

We can write this result in terms of the off-axis k-vector components:

$E(x,y)$ = const if a plane wave

$$E\left(k_x,k_y\right) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-i\left(k_x x + k_y y\right)\right] t(x,y)\, E(x,y)\, dx\, dy$$

Aperture transmission function

that is:

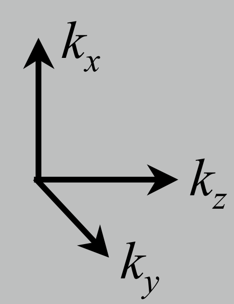
$$E\left(k_x,k_y\right) \propto \mathscr{F}\left\{t(x,y)\,E(x,y)\right\}$$

and:

$$k_x = kx_1/z \quad \text{and} \quad k_y = ky_1/z$$

or:

$$\theta_x = k_x/k = x_1/z \quad \text{and} \quad \theta_y = k_y/k = y_1/z$$



The Uncertainty Principle in Diffraction!

$$E\left(k_x, k_y\right) \propto \mathscr{F}\left\{t(x,y)\,E(x,y)\right\} \qquad k_x = kx_1/z$$

Because the diffraction pattern is the Fourier transform of the slit, there's an uncertainty principle between the slit width and diffraction pattern width!

If the input field is a plane wave and Δx is the slit width and Δk_x is the proportional to the beam angular width after the screen,

$$\Delta x\, \Delta k_x > 1$$

Or:

$$\Delta x\, \Delta x_1 > z / k$$

The smaller the slit, the larger the diffraction angle and the bigger the diffraction pattern!

Fraunhofer Diffraction from a slit

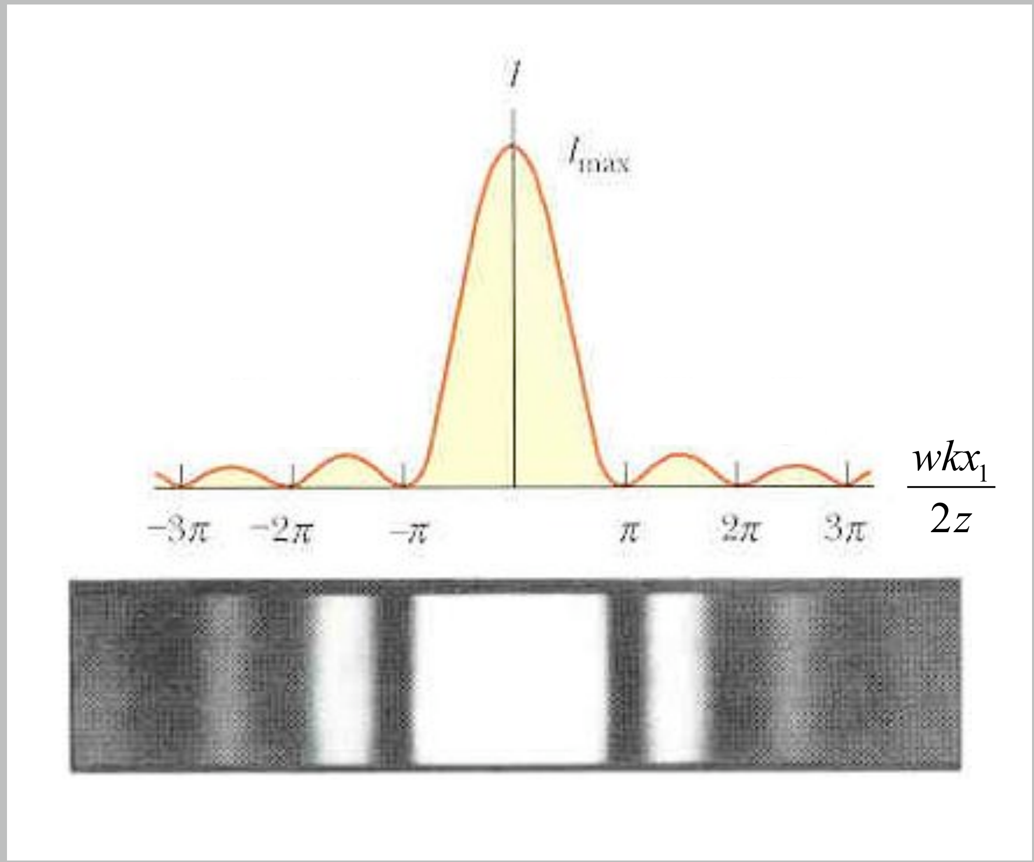
Fraunhofer Diffraction from a slit is simply the Fourier Transform of a rect function, which is a sinc function. The irradiance is then sinc^2.

$$t(x) = \mathrm{rect}[x/w]$$

$$E(k_x) \propto \mathscr{F}\{t(x)\}$$

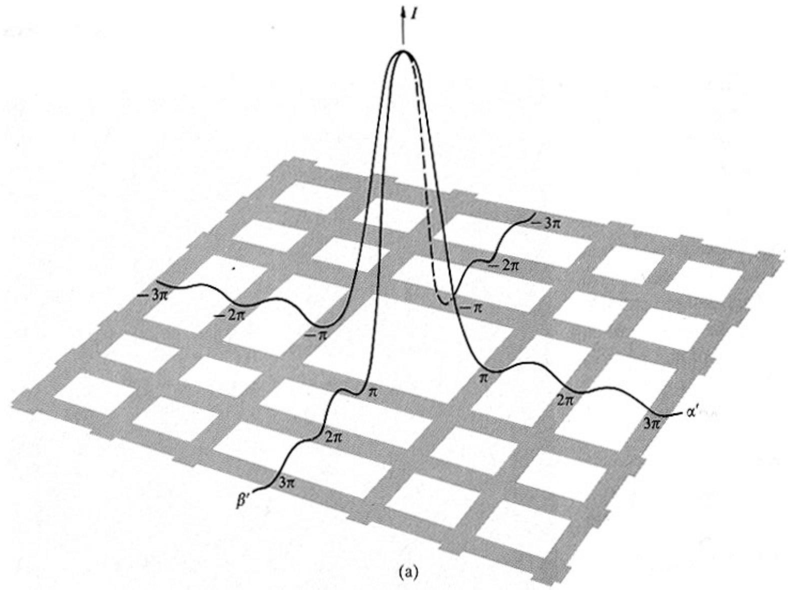
$$\boxed{\begin{aligned} E(k_x) &\propto \mathrm{sinc}(wk_x/2) \\ E(x_1) &\propto \mathrm{sinc}(wkx_1/2z) \end{aligned}}$$

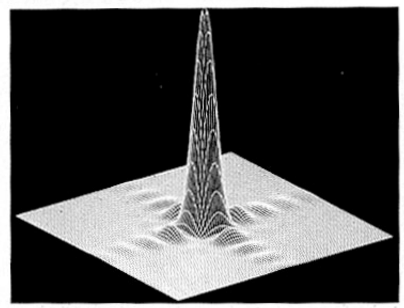
$$\boxed{\begin{aligned} I(k_x) &\propto \mathrm{sinc}^2(wk_x/2) \\ I(x_1) &\propto \mathrm{sinc}^2(wkx_1/2z) \end{aligned}}$$



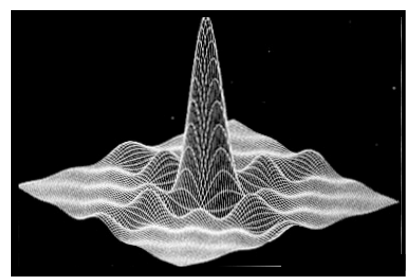
Fraunhofer Diffraction from a Square Aperture

The diffracted field is a sinc function in both x_1 and y_1 because the Fourier transform of a rect function is sinc.





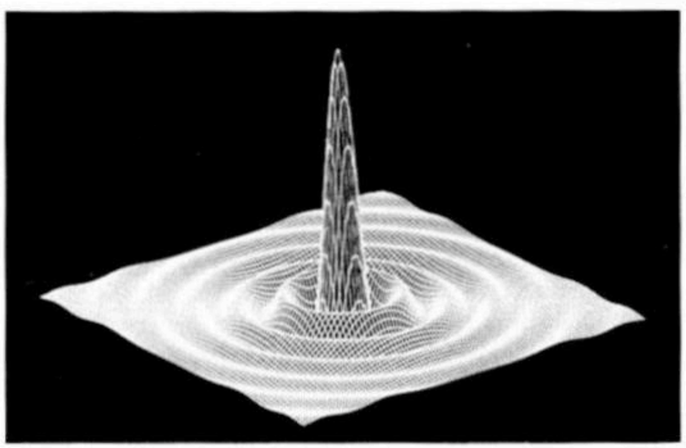
Diffracted irradiance



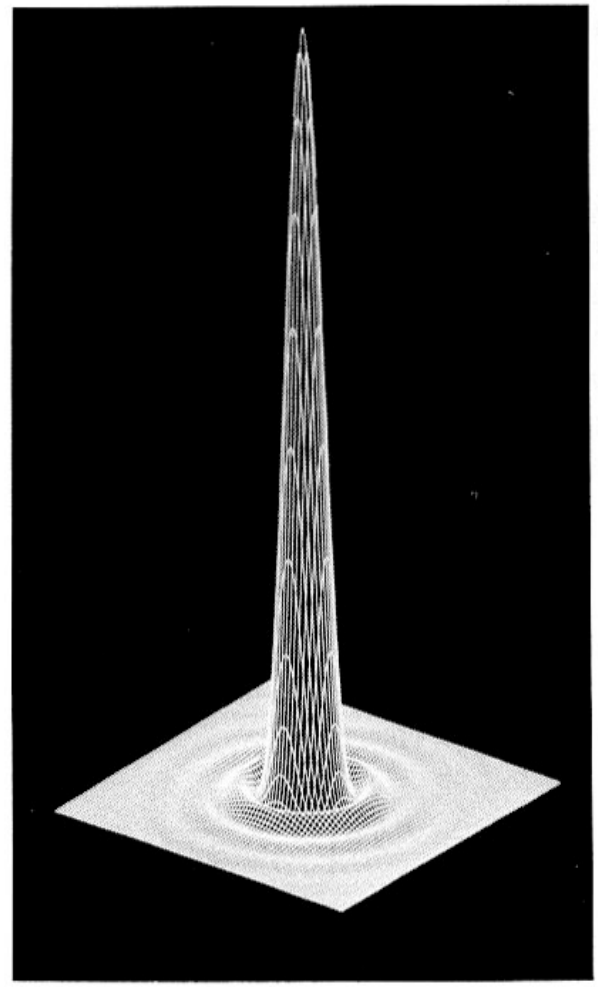
Diffracted field

Diffraction from a Circular Aperture

A circular aperture yields a diffracted "Airy Pattern," which looks a lot like a sinc function, but actually involves a Bessel function.

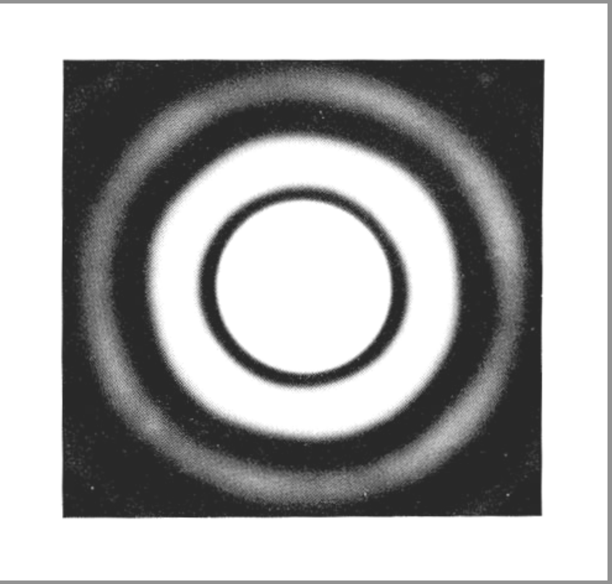


Diffracted field



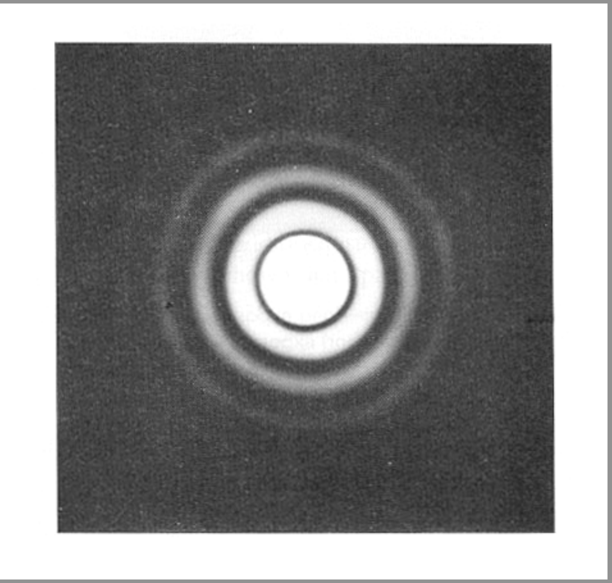
Diffracted Irradiance

Diffraction from small and large circular apertures



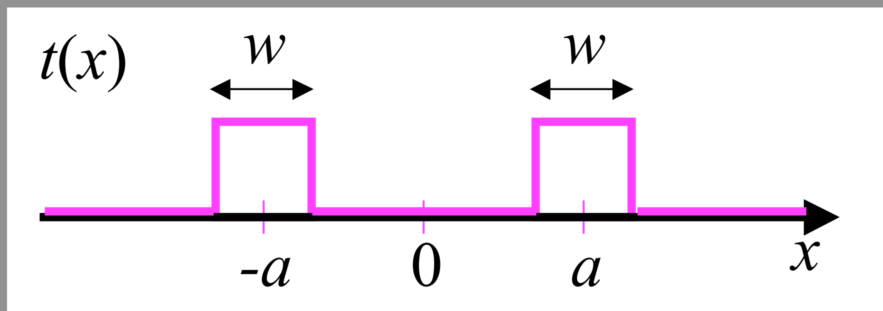
Far-field intensity pattern from a small aperture

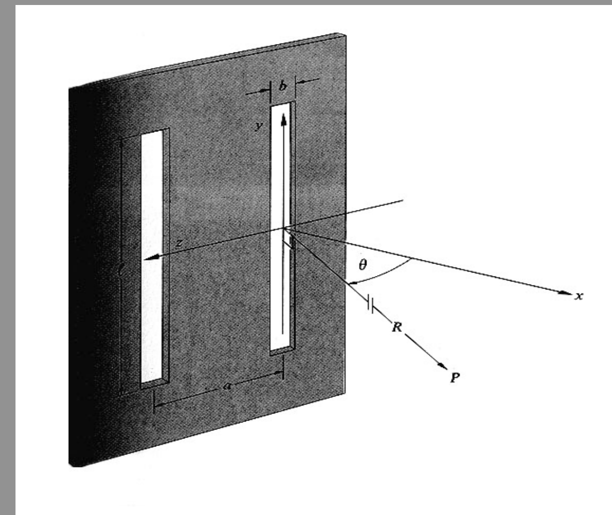
Recall the Scale Theorem! This is the Uncertainty Principle for diffraction.



Far-field intensity pattern from a large aperture

Fraunhofer diffraction from two slits



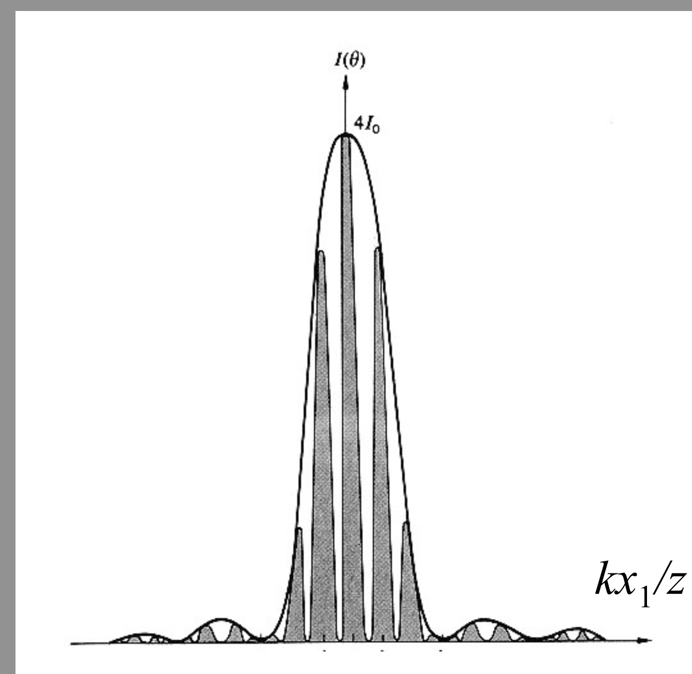


$$t(x) = \mathrm{rect}[(x+a)/w] + \mathrm{rect}[(x-a)/w]$$

$$E(k_x) \propto \mathscr{F}\{t(x)\}$$

$$\propto \mathrm{sinc}[w(kx_1/z)/2]\exp[+ia(kx_1/z)] + \mathrm{sinc}[w(kx_1/z)/2]\exp[-ia(kx_1/z)]$$

$$\boxed{E(x_1) \propto \mathrm{sinc}(wkx_1/2z)\cos(akx_1/z)}$$

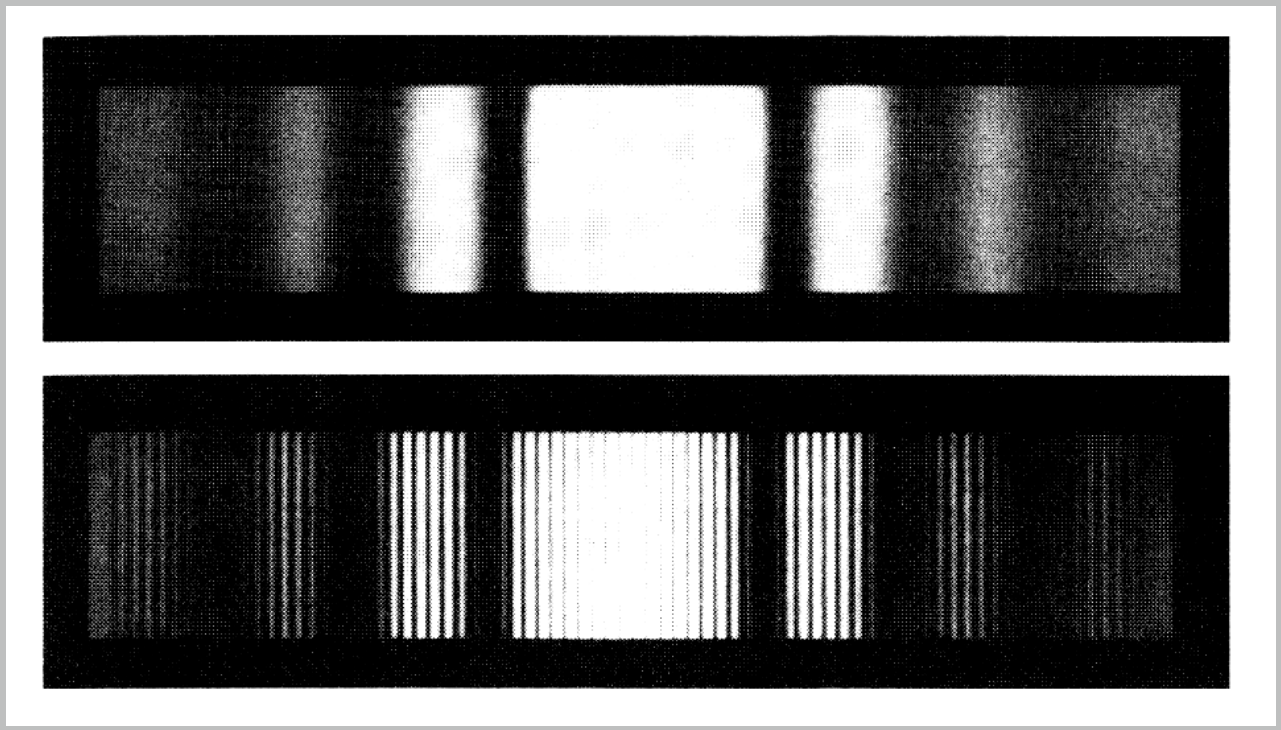


Diffraction from one- and two-slit screens

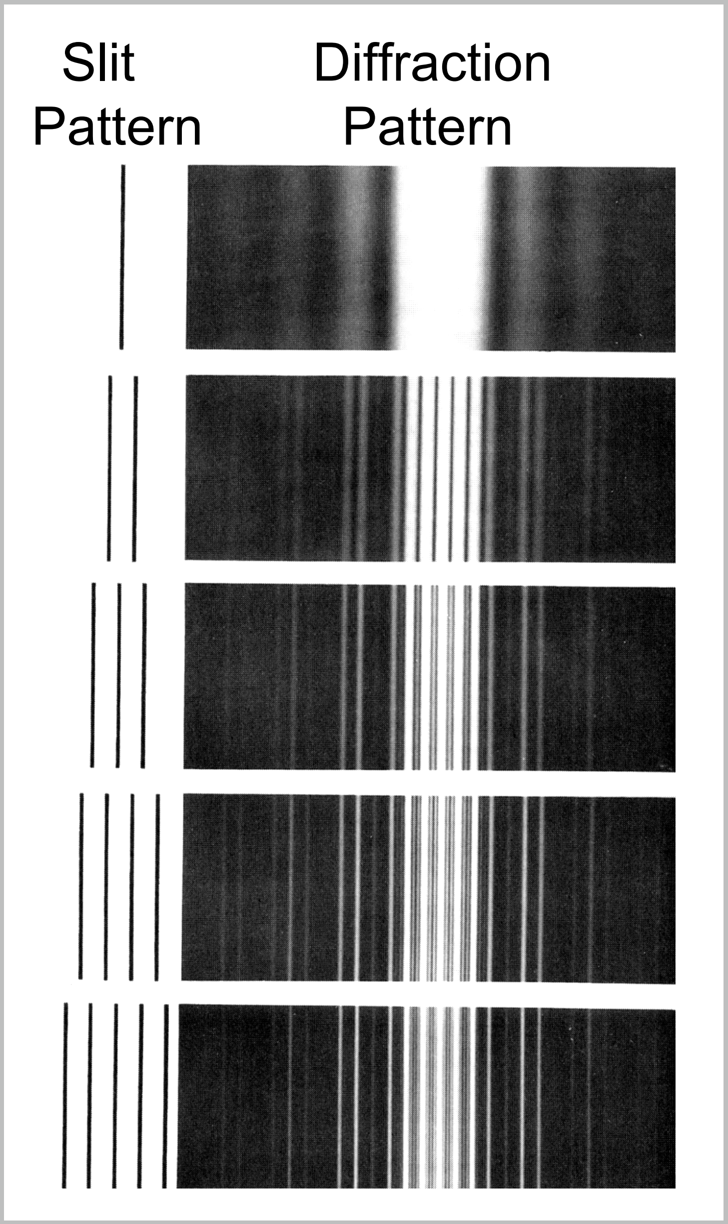
Fraunhofer diffraction patterns

One slit

Two slits

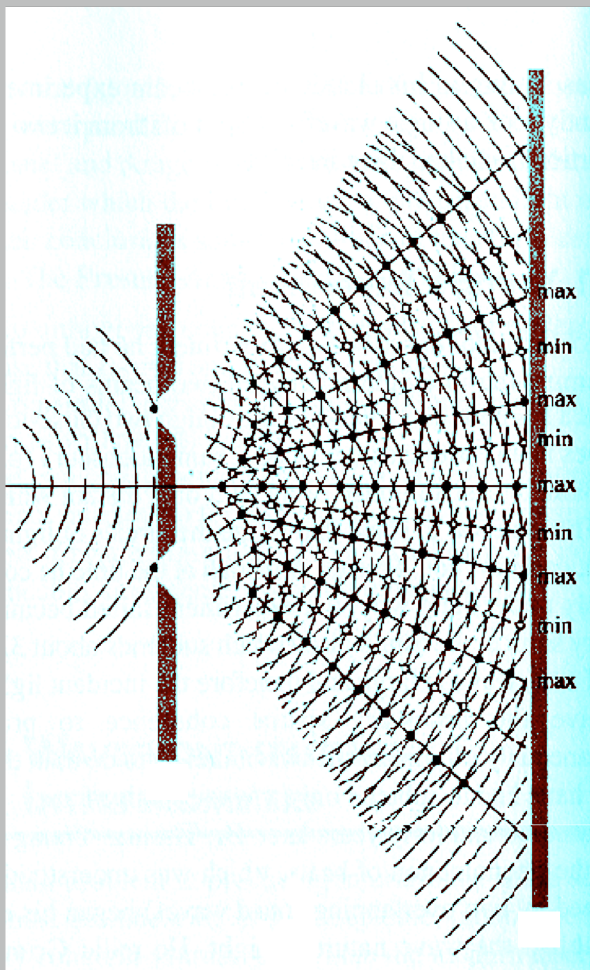


Diffraction from multiple slits



Infinitely many equally spaced slits (a Shah function!) yields a far-field pattern that's the Fourier transform, that is, the Shah function.

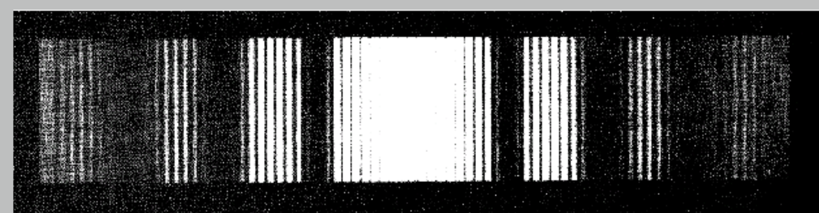
Two Slits and Spatial Coherence



If the spatial coherence length is less than the slit separation, then the relative phase of the light transmitted through each slit will vary randomly, washing out the fine-scale fringes, and a one-slit pattern will be observed.

Fraunhofer diffraction patterns

Good spatial coherence



Poor spatial coherence

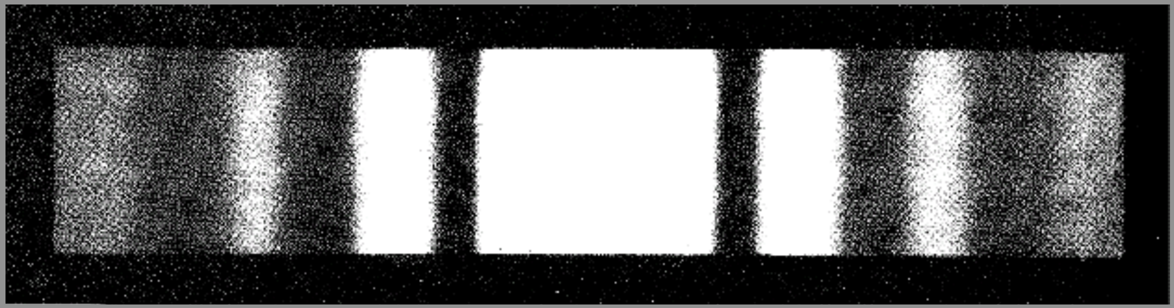
Young's Two Slit Experiment and Quantum Mechanics

Imagine using a beam so weak that only one photon passes through the screen at a time. In this case, the photon would seem to pass through only one slit at a time, yielding a one-slit pattern.

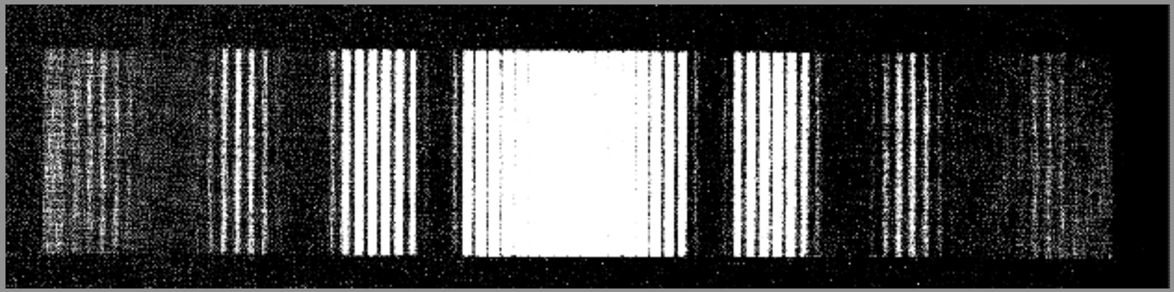
Which pattern occurs?

Possible Fraunhofer diffraction patterns

Each photon passes through only one slit

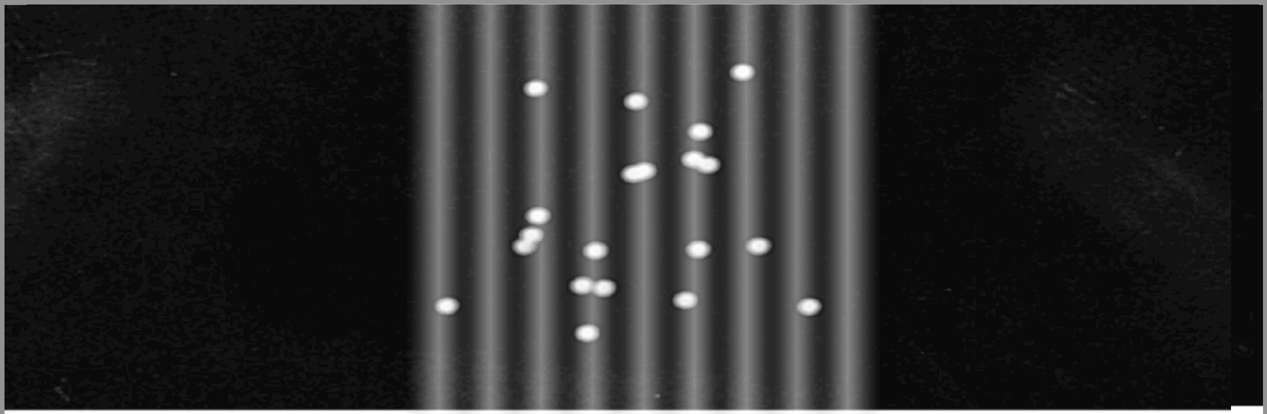


Each photon passes through both slits



Dimming the light incident on two slits

Dimming the light in a two-slit experiment yields single photons at the screen. Since photons are particles, it would seem that each can only go through one slit, so then their pattern should become the single-slit pattern.



(a) 20 counts

Each individual photon goes through both slits!