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Cooley J W & Tukey J W. An algorithm for the machine calculation of complex Fourier series. *Math. Comput.* 19:297-301, 1965.
[IBM Watson Research Center, Yorktown Heights, NY; Bell Telephone Laboratories, Murray Hill; and Princeton University, NJ]

This paper, on the fast Fourier transform algorithm, was at first credited with a great discovery. A numberof strange and fortuitous circumstances led to its publicity and to the failure of earlier discoveries to achieve recognition and acceptance. [The *SCI*® indicates that this paper has been cited in more than 2,155 publications, making it the most-cited paper published in this journal.]

On the Origin and Publication of the FFT Paper

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During a meeting of President Kennedy's Scientific Advisory Committee, sometime in 1963, Dick Garwin (then at the Columbia University IBM Watson Research Center, New York City) noticed that John Tukey (then professor of mathematics, Princeton University), who sat next to him, was, in his usual way, doodling with a notepad, jotting down Fourier transform formulas, a subject which Dick was very interested in. When asked, John said that he was working on a better algorithm for computing the discrete Fourier transform (DFT). He expressed the DFT in the form

$$a(k) = \sum_{j=0}^{N-1} x(j) W^{jk},$$

where W= exp(2πi/N) and k = 0,1,...,N-1. This is a discrete representation of a Fourier series or a Fourier integral at equally spaced points. He showed that if N is a product, say *N-ab,* one can express the series as an a-point series of o-point subseries. This would reduce the number of multiplications and additions for complex data from N^2 to N(a+b). (To be more precise, the direct method, according to the above formula, would take N^2 complex multiplications and N(AM) complex additions.) He added that this may be repeated for more factors of *N* and that if *N* is a power of two, one can obtain an algorithm taking $N\log_2 N$ operations. (If the data are complex, this is actually $.5N\log_2 N$complex multiplications and $N\log_2 N$ complex additions.)

Dick realized that this was very important and from that time on, he put an enormous amount of energy into developing and publicizing the algorithm. He went to Bill Dorn, manager of the computing center at the IBM T.J. Watson Research Center, with his notes from his conversation with John. Bill brought Dick to Jim Cooley and asked him to work on it. Dick said he needed the fast Fourier transform (FFT) to compute a three-dimensional Fourier transform of spin orientations of He3.

Not realizing the importance of the algorithm, Jim did not give it a high priority until he got some phone calls from Dick. Jim derived a radix-2 algorithm, which saved memory and address generation by overwriting data with results, and wrote a program for doing one-, two-, and three-dimensional FFTs for *N*equal to a power of two. He sent the program to Dick and an increasing number of others to whom he had energetically been spreading the good word.

Actually, Dick was aware of a very great need for the FFT in a broad range of applications and had greater ambitions for it. It was revealed later[1] that an immediate and important need arose in Dick's studies of the possibilities for a nuclear test ban treaty. Since Russia would not agree at that time to on-site inspections, Dick was studying the feasibility of detecting nuclear explosions by means of spectral analyses of data from seismic sensing devices. He realized that the main obstacle was in the amount of computing of Fourier transforms that would have to be done.

Jim described the FFT in a weekly numerical analysis seminar conducted by Larry Horowitz in the IBM Research Department of Mathematical Sciences. Ken Iverson and other APL designers were there, and, for their presentation, Howard Smith wrote an APL program for the FFT. (This was before APL was implemented on any computer.) These talks gave the FFT a thorough exposure to a number of mathematicians. Frank Thomas, a mathematically trained patent attorney participating in the seminar, suggested that there might be patent possibilities. A meeting of several mathematicians and attorneys was called to discuss the patentability of the FFT algorithm. They asked Jim to call John to find out about the origins of his ideas for the algorithm. He did so, and John mentioned work by F. Yates[2] that contained a doubling algorithm for a calculation which was similar to what were later called Walsh and Hadamard transforms. He also mentioned I.J. Good,3.[4] who invented a class of algorithms very similar to the FFT algorithm, except that they required that the factors of A/be mutually prime. Thus, there appeared to be no previous use of the FFT algorithm. After considering a number of suggestions, it was finally agreed that the algorithm should be put in the public domain to prevent anyone from patenting it. This was done by having Shmuel Winograd and Ray Miller design a device for calculating with the FFT algorithm and having Jim write a paper with a footnote mentioning what they did. Jim put his notes together, including a description of the general arbitrary radix algorithm, and sent a draft of the article to John, asking him to go over it and be a coauthor. He did so, and around August 1964 this *Citation Classic*® paper was sent to Eugene Isaccson, an editor of *Mathematics of Computation.* It was published in the April 1965 issue.

While all this was going on, John outlined his method in a spring 1963 course at Princeton University, where Gordon Sande, then a graduate student, wrote a radix-2 FFT program and a paper. He went to John with the article to discuss publication but withdrew it when he found that our paper had already been submitted. It is not clear why he did not publish it, since he included methods for computing correlations, which are important applications for the FFT, and used a different, closely related algorithm. Instead, Gordon and Morven Gentleman gave an. excellent and comprehensive paper at the 1966 Fall Joint Computer Conference,[5] in which they not only covered all that was mentioned above, but also did a Wilkinson-type error analysis showing that the improvement of the FFT in accuracy was in the same ratio as the improvement in speed. They also showed how very large FFTs could be done with limited high speed memory and auxiliary storage. These methods are still being used in supercomputers today.

At about the same time, Dick asked his colleague, Llewellyn Thomas, at the Columbia University IBM Watson Laboratory, about the FFT. Thomas said that he had used and published the prime factor algorithm[6] after looking in the library for a better way to compute Fourier transforms on an IBM tabulating machine. He found the factoring ideas in books by Karl Stumpff,[7,8] who referred to earlier work by C. Runge and H. Konig.[9] Earlier publications of the "method of subseries" have turned up many times. Herman Goldstine[10] found the earliest known reference in an obscure paper by C.F. Gauss[1]1 that had not been translated from its original Latin. This was the inspiration for a thorough history of the FFT by M.T. Heideman, D.H. Johnson, and C.S. Burrus.[12]

It may seem surprising that the basic ideas of the FFT existed so long unexploited when they could have been used to very great advantage. One explanation becomes apparent when reading the works of Cornelius Lanczos,[13]-[14] who described the doubling algorithm in 1942 not only for its efficiency, but also as a method for checking accuracy during calculation. He later published two books on Fourier analysis mentioning the doubling algorithm only once (in a footnote). The explanation seems to be that for the sizes of *N* used in those days, many other algorithms, usually using the symmetries and trigonometric relations of the sinusoids, were more efficient. It was only when electronic computers made computing with large Ws possible that the FFT became more efficient. Then, perhaps by chance, the "method of subseries" went unnoticed while speeds and capacities of computers grew well beyond the point where the FFT could first be of value. It makes one wonder if there are more gems of ideas in those old volumes left by the masters of computing in the days of hand calculations.

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