M.L. Frame & B.B. Mandelbrot
Fractals, Graphics, & Mathematics Education
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Foreword by Lynn A. Steen

In 1980 Ronald Reagan was elected president of the United States, IBM began an urgent program to develop a personal computer for which Microsoft agreed to provide the operating system, and Benoit Mandelbrot got his first real look at the archetypal fractal, the eponymous Mandelbrot set. One hundred years from now, which of these events is more likely to be remembered as having had the greatest influence on science and human affairs?

At this moment, two decades later, personal computers seem to be well in the lead. PCs are on every desk, all now connected by the world-linking Internet. The personal computer has truly transformed the way the world works. No invention since the printing press has created such a widespread impact and no human activity has ever changed society so quickly.

However, when we examine Internet patterns in detail, guess what we find just beneath the surface? The footprints of fractals. These wondrous geometric objects, discovered by Mandelbrot just over twenty years ago, turn out to be the key to understanding the frenetic behavior of signals linking the world's computers, as well as the means to efficient compression that makes it possible to transmit images over the Internet. Without fractals, engineers would never have been able to make the World Wide Web work as well as it does. And without the Web, PCs would be just one more labor-saving appliance.

Computers are important instrumentally; they provide tools that enable us to work more efficiently, to see patterns previously hidden, and to organize information in new and revealing ways. In contrast, fractals are important fundamentally; they provide elements of a totally new geometry that offer a profoundly different way to understand nature. In the long run, this new understanding of nature will count for far more than momentary advances in technology or politics.

The predominant Western view of the relation between mathematics and nature is a legacy of Plato's distinction between a world of ideals and a world of actualities. Mathematics, in this view, belongs to the ethereal world of ideals; nature, being earthly rather than heavenly, belongs to a world of actualities that is both imperfect and incomplete. Thus, reality is best understood by approximation in terms of ideal mathematical models.

Our most important inheritance from this tradition is Euclidean geometry, the axiomatic study of lines, circles, and triangles that form an ideal (and therefore approximate) basis for understanding geography, mechanics, astronomy, and everything real. From this perspective, nature is like noisy mathematics—rough and crumpled, slightly out of focus.

Fractals create an alternative to Euclidean geometry whose elements are not lines and circles but iterations and self-similarities, whose surfaces are not smooth but jagged, whose features are not perfect but broken. Derived from apparent pathologies that puzzled or affronted traditional mathematicians, fractals reveal an entirely new geometry that enables us to understand formerly inexplicable real-world phenomena.

Fractals provide insight into the distribution of galaxies, the spread of bacterial colonies, the grammar of DNA, the shape of coastlines, changes in climate, development of hurricanes, growth of crystals, percolation of ground pollutants, turbulence of fluids, and the path of lightning. They have been employed to create powerful antennas, to develop fiber optics, to monitor financial data, to compress images, and to produce artificial landscapes. Their influence has been felt in art, architecture, drama (*Arcadia*), film (*Jurassic Park*), music, and poetry. Fractals have even penetrated the inner sanctum of elite culture—New Yorker cartoons.

For many, this extraordinary utility would be a sufficient warrant for fractals to be awarded a prominent role in mathematics education. But reasons other than utility can also be advanced, and it is these reasons, not utility, that form the major thrust of this volume.

Simply put, fractals enable everyone to enjoy mathematics. Nothing else can make such a striking—and important—claim.

Arguments about strategies for improving mathematics education roil state and local school politics. Some urge strict exam-enforced standards; others advocate more inviting contexts for learning. Some promote traditional curricula; others support enhanced or integrated programs. Rarely do the protagonists in these "math wars" stop to ask whether different mathematics might yield increased learning.

But that is precisely the argument advanced by the authors represented in this volume. They focus on teaching fractals, not primarily because fractals are important but because learning about fractals is, as one student put it, "indescribably exciting . . . and uniquely intriguing." It is easy to see why:

- The first steps are so much fun. Exploring fractals creates unprecedented enthusiasm for discovery learning among both teachers and students.

- Fractals are beautiful. Stunning visuals appeal to the mind's eye and create contagious demand for continued exploration.

- Anyone can play. Exploration of fractal geometry appeals to students of every age, from primary school through college and beyond.

- Fractals promote curiosity. Simple rules, easily modified, create nearly uncontrollable temptations to explore different options to see what surprising patterns will emerge.

- Simple ideas lead to unexpected complexity. Fractals are more life-like than objects studied in other parts of mathematics; thus they appeal to many students who find traditional mathematics cold and austere.

- Many easy problems remain unsolved. Fractals are rich in open conjectures that lead to deep mathematics. Moreover, the distance from elementary steps to unsolved problems is very short.

- Careful inspection yields immediate rewards. Insight and conjectures arise readily when our well-developed visual intuition is applied to fractal images. In studying fractals, children can see and conjecture as well as adults.

- Computers enhance learning. The visual impact of computer graphics makes fractal images unforgettable, while the unforgiving logical demands of computer programs yield important lessons in the value of rigorous thinking.

The history of mathematics education is long and convoluted, reflecting both the changing nature of mathematics and the evolving demands of society. Although in the eighteenth and nineteenth centuries mathematics was both experimental and theoretical, during much of the twentieth century the theoretical aspect has dominated. Much of mathematics education followed this trend towards theory and abstraction, leading to alarming reports of rising mathematical illiteracy not only in the United States but in many other countries as well.

Fractals represent a rebirth of experimental mathematics, enabled by computers and enhanced by powerful evidence of utility. In the ebb and flow of mathematical fashion, the struggle between theoretical and experimental is once again more nearly in balance. What remains is the challenge of restoring this balance to mathematics education. It is to that important task that this book is devoted.

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Chapter 1

Some Reasons for the Effectiveness of Fractals in Mathematics Education

Benoit B. Mandelbrot and Michael Frame

Short is the distance between the elementary and the most sophisticated results, which brings rank beginners close to certain current concerns of the specialists. There is a host of simple observations that everyone can appreciate and believe to be true, but not even the greatest experts can prove or disprove. There is a supply of unsolved, elementary problems that give students the opportunity to learn how mathematics can be done by enabling them to do new (if not necessarily earth-shaking) mathematics; there is a continuing flow of new results in unexpected directions.

1 Introduction

In the immediate wake of Mandelbrot (1982), fractals began appearing in mathematics and science courses, mostly at the college level, and usually in courses on topics in geometry, physics, or computer science. Student reaction often was extremely positive, and soon entire courses on fractal geometry (and the related discipline of chaotic dynamics) arose. Most of the initial offerings were aimed at students in science and engineering, and occasionally economics, but, something about fractal geometry resonated for a wider audience. The subject made its way into the general education mathematics and science curriculum, and into parts of the high school curriculum. Eventually, entire courses based on fractal geometry were developed for humanities and social sciences students, some fully satisfy the mathematics or science requirement for these students. As an introduction to this volume, we share some experiences and thoughts about the effectiveness and appropriateness of these courses.

As teachers, we tell our students to first present their case and allow the objections to be raised later by the devil's advocate. But we decided to preempt some of the advocate's doubts or objections before we move on with our story.

1.1 The early days

A few years ago, the popularity of elementary courses using fractals was largely credited to the surprising beauty of fractal pictures and the centrality of the computer to instruction in what lies behind those pictures. A math or science course filled with striking, unfamiliar visual images, where the computer was used almost every day, sometimes by the students? The early general education fractals courses did not fit into the standard science or mathematics format, a novel feature that contributed to their popularity.

1.2 What beyond novelty?

We shall argue that novelty was neither the only, nor the most significant factor. But even if it had been, and if the popularity of these courses had declined as the novelty wore off, so what? For a few years we would have had effective vehicles for showing a wide audience that science is an ongoing process, an exciting activity pursued by living people. While introductory courses for majors are appropriate for some non-science students, and qualitative survey courses are appropriate for some others, fractal geometry provided a middle ground between quantitative work aiming toward some later reward (only briefly glimpsed by students not going beyond the introductory course), and qualitative, sometimes journalistic, sketches. In general education fractal geometry courses, students with only moderate skills in high school algebra could learn to do certain things themselves rather than read forever about what others had done. They could grow fractal trees, understand the construction of the Mandelbrot and Julia sets, and synthesize their own fractal mountains and clouds. Much of this mathematics spoke directly to their visible world. Many came away from these courses feeling they had understood some little bit of how the world works. And the very fact that some of the basic definitions are unsettled, and that there are differences of opinion among leading players, underscored the human aspect of science. No longer a crystalline image of pure deductive perfection, mathematics is revealed to be an enterprise as full of guesses, mistakes, and luck as any other creative activity. Even if the worst fears had been fulfilled, we would have given several years of humanities and social science students a friendlier view of science and mathematics.

Fortunately, anecdotal evidence suggests that, while much of the standard material and computerized instruction techniques are no longer novel, the audience for fractal geometry courses is not disappearing, thus disproving those fears.

1.3 What aspects of novelty have vanished?

Success destroyed part of the novelty of these courses. Now images of the Mandelbrot set appear on screen savers, T-shirts, notebooks, refrigerator magnets, the covers of books (including novels), MTV, basketball cards, and as at least one crop circle in the fields near Cambridge, UK. Fractals have appeared in novels by John Updike, Kate Wilhelm, Richard Powers, Arthur C. Clarke, Michael Crichton, and others. Fractals and chaos were central to Tom Stoppard's play *Arcadia*, which includes near quotes from Mandelbrot. Commercial television ("Murphy Brown," "The Simpsons," "The X-Files"), movies ("Jurassic Park"), and even public radio ("A Prairie Home Companion") have incorporated fractals and chaos. In the middle 1980s, fractal pictures produced "oohhs," "aahhs," and even stunned silence; now they are an ingrained part of both popular and highbrow culture (the music of Wuorinen and Ligeti, for example). While still beautiful, they are no longer novel.

A similar statement can be made about methodology. In the middle 1980s, the use of computers in the classroom was uncommon, and added to the appeal of fractal geometry courses. Students often lead faculty in recognizing and embracing important new technologies. The presence of computers was a definite draw for fractal geometry courses. Today, a randomly selected calculus class is reasonably likely to include some aspect of symbolic or graphical computation, and many introductory science classes use computers, at least in the lab sections. The use of computers in many other science and mathematics courses no longer distinguishes fractal geometry from many other subjects.

1.4 Yet these courses' popularity survived their novelty. Why is this?

Instead of being a short-lived fad, fractal geometry survived handsomely and became a style, part of our culture.

The absence of competition is one obvious reason: fractal geometry remains the most visual subject in mathematics and science. Students are increasingly accustomed to thinking pictorially (witness the stunning success of graphical user interfaces over sequences of command lines) and continue to be comfortable with the reasoning in fractal geometry. Then, too, in addition to microscopically small and astronomically large fractals, there is also an abundance of human-sized fractals, whereas there are not human-sized quarks or galaxies.

Next, we must mention surprises. Students are amazed the first time they see that for a given set of rules, the deterministic IFS algorithm produces the same fractal regardless of the starting shape. The gasket rules make a gasket from a square, a single point, a picture of your brother, . . . anything. If the Mandelbrot set is introduced by watching videotapes of animated zooms, then the utter simplicity of the algorithm generating the Mandelbrot set is amazing. Part of what keeps the course interesting is the surprises waiting around almost every corner. Also, besides science and mathematics, fractals have direct applications in many fields, including music, literature, visual art, architecture, sculpture, dance, technology, business, finance, economics, psychology, and sociology. In this way, fractals act as a sort of common language, *lingua franca*, allowing students with diverse backgrounds to bring these methods into their own worlds, and in the context of this language, better understand some aspects of their classmates' work.

Three other reasons are more central to the continued success of general education fractal geometry courses. By exploiting these reasons, we keep strengthening current courses and finding directions for future development.

As a preliminary, let us briefly list these reasons for the pedagogical success of fractal geometry. We shall return to each in detail.

1.4.1 First, a short distance from the downright elementary to the hopelessly unsolved

First surprise: truly elementary aspects of fractal geometry have been successfully explained to elementary school students, as seen in Chapters 10 and 13. From those aspects, there is an uncannily short distance to unsolved problems. Few other disciplines—knot theory is an example—can make this claim.

Many students feel that mathematics is an old, dead subject. And why not? Most of high school mathematics was perfected many centuries ago by the Greeks and Arabs, or at the latest, a few centuries ago by Newton and Leibnitz. Mathematics appears as a closed, finished subject. To counter that view, nothing goes quite so far as being able to understand, after only a few hours of background, problems that remain unsolved today. Number theory had a standard unsolved but accessible problem that need not be named. Alas, that problem now is solved. Increasing our emphasis on unsolved problems brings students closer to an edge of our lively, growing field and gives them some real appreciation of science and mathematics as ongoing processes.

1.4.2 Second, easy results remain reachable

The unsolved problems to which we alluded above are very difficult, and have been studied for years by experts. In contrast, not nearly all the easy aspects of fractal geometry have been explored. At first, this may seem more relevant to graduate students, but in fact, plenty of the problems are accessible to bright undergraduates. The National Conference of Undergraduate Research and the Hudson River Undergraduate Mathematics Conference, among others, include presentations of student work on fractal geometry. It may be uncommon for students in a general education course to make new contributions to fractal geometry (though to be sure they often come up with very creative projects applying fractal concepts to their own fields), but their classmates in sciences and mathematics can and do. (See Frame & Lanski (1999).) Incorporating new work done by known, fellow undergraduates can have an electrifying effect on the class. Few things bring home the accessibility of a field so much as seeing and understanding something new done by someone about the same age as the students. Then, too, this is quite exciting for the science and mathematics students whose work is being described. And it can be, and has been, a catalyst for communication between science and non-science students. So far as we know, in no other area of science or mathematics are undergraduates so likely to achieve a sense of ownership of material.

1.4.3 Third, new topics continue to arise and many are accessible

New things, accessible at some honest level, keep arising in fractal geometry. Of course, new things are happening all around, but the latest advances in superstring theory, for example, cannot be described in any but the most superficial level in a general education science course. This is not to say all aspects of fractal geometry are accessible to nonspecialists. Holomorphic surgery, for instance, lives in a pretty rarefied atmosphere. And there is deep mathematics underlying much of fractal geometry. But pictures were central to the birth of the field, and most open problems remain rooted in visual conjectures that can be explained and understood at a reasonable level without the details of the supporting mathematics. While undergraduates can do new work, it is unlikely to be deep work. In fractal geometry much of even the current challenging new work can be presented only in part but, honestly, and without condescension to our students.

Later we shall further explore some aspects of each of these points.

1.5 Most important of all: curiosity

Teaching endless sections of calculus, precalculus, or baby statistics to uninterested audiences is hard work and all too often we yield to the temptation to play to the lowest third of the class. The students merely try to survive their mathematics requirement. Little surprise we complain about our students' lack of interest, and about the disappearance of childlike curiosity and sense of wonder.

Fractal geometry offers an escape from this problem. It is risky and doesn't always work, for it relies on keeping this youthful curiosity alive, or reawakening it if necessary. In the final *Calvin and Hobbes* comic strip, Calvin and Hobbes are on a sled zipping down a snow-covered hill. Calvin's final words are, "It's a magical world, Hobbes ol' buddy. Let's go exploring!" This is the feeling we want to awaken, to share with our students.

Teaching in this way, especially emphasizing the points we suggest, demands faith in our students. Faith that by showing them unsolved problems, work done by other students, and new work done by scientists, they will respond by accepting these offerings and becoming engaged in the subject. It does not always work. But when it does, we have succeeded in helping another student become a more scientifically literate citizen. Surely, this is a worthwhile goal.

2 Instant gratification: from the elementary to the diabolic and unsolved, the shortest distance is...

In most areas of mathematics, or indeed of science, a vast chasm separates the beginner from even understanding a statement of an unsolved problem. The Poincarè conjecture is a very long way from a first glimpse of topological spaces and homotopies. Science and mathematics courses for non-majors usually address unsolved problems in one of two ways: complete neglect or vast oversimplification. This can leave students with the impression that nothing remains to be done, or that the frontiers are far too distant to be seen; neither picture is especially inviting.

Fractal geometry is completely different. While the *solutions* of hard problems often involve very clever use of sophisticated mathematics, frequently the *statements* do not. Here we mention two examples, to be amplified and expanded on in the next chapter.

The first observed example of Brownian motion occurred in a drop of water: pollen grains dancing under the impact of molecular bombardment. Nowadays this can be demonstrated in class with rather modest equipment: a microscope fitted with a video camera and a projector. Increasing the magnification reveals ever finer detail in the dance, thus providing a visual hint of self-similarity. A brief description of Gaussian distributions—or even of random walk—is all we need to motivate computer simulations of Brownian motion. Taking a Brownian path for a finite duration and subtracting the linear interpolation from the initial point to the final point produces a Brownian plane cluster. The periphery, or *hull*, of this cluster looks like the coastline of an island. Together with numerical experiments, this led to the conjecture that the hull has dimension $4/3$. Dimension is introduced early in fractal geometry classes, so freshman English majors can understand this conjecture. Yet it is unproved.[1]

No icon of fractal geometry is more familiar than the Mandelbrot set. Its strange beauty entrances amateurs and experts alike. Many credit it with the resurgence of interest in complex iteration theory, and its role in the birth of computer-aided experimental mathematics is incalculable. For students, the first surprise is the simplicity of the algorithm to generate it. For each complex number c, start with $z_0 = 0$ and produce the sequence $z_1, z_2, \ldots$ by $z_{i+1} = z_i^2 + c$. The point c belongs to the Mandelbrot set if and only if the sequence remains bounded. How can such a simple process make such an amazing picture? Moreover, a picture that upon magnification reveals an infinite variety of patterns repeating but with variations. One way for the sequence to remain bounded is to converge to some repeating pattern, or cycle. If all points near to $z_0 = 0$ produce sequences converging to the same cycle, the cycle is stable. Careful observation of computer experiments led Mandelbrot to conjecture that arbitrarily close to every point of the Mandelbrot set lies a c for which there is a stable cycle. All of these concepts are covered in detail in introductory courses, so here, too, beginning students can get an honest understanding of this conjecture, unsolved despite heroic effort.

3 Some easy results remain: "There's treasure everywhere"

3.1 Discovery learning

Learning is about discovery, but undergraduates usually learn about past discoveries from which all roughness has been polished away giving rise to elegant approaches. Good teaching style, but also speed and efficiency, lead us to present mathematics in this fashion. The students' act of discovery dissolves in becoming comfortable with things already known to us. Regardless of how gently we listen, this is an asymmetric relationship: we have the sought-after knowledge. We are the masters, the final arbiters, they the apprentices.

In most instances this relationship is appropriate, unavoidable. If every student learned mathematics and science by reconstructing them from the ground up, few would ever see the wonders we now treasure. Which undergraduate would have discovered special relativity? But for most undergraduate mathematics and science students, and nearly all non-science students, this master-apprentice relationship persists through their careers, leaving no idea of how mathematics and science are done. Fractal geometry offers a different possibility.

Term projects are a central part of our courses for both non-science and science students. To be sure, some projects turn out less appropriate than hoped, but many have been quite creative. Refer to the *student project* entries in *A Guide to the Topics.* Generally, giving a student an open-ended project and the responsibility for formulating at least some of the questions, and being interested in what the student has to say about these questions, is a wonderful way to extract hard work.

3.2 A term project example: connectivity of gasket relatives

We give one example, Kern (1997), a project of a freshman in a recent class. Students often see the right Sierpinski gasket as one of the first examples of a mathematical fractal. The IFS formulation is especially simple: this gasket is the only compact subset of the plane left invariant by the transformations

$$T_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right),$$
$$T_2(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right) + \left(\frac{1}{2}, 0\right),$$
$$T_3(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right) + \left(0, \frac{1}{2}\right).$$

Applying these transformations to the unit square $S = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ gives three squares $S_i = T_i(S)$ for $i = 1, 2, 3$. Among the infinitely many changes of the T_i, in general producing different fractals, a particularly interesting and manageable class consists of including reflections across the x- and y-axes, rotations by $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$, and appropriate translations so the three resulting squares occupy the same positions as $T_1(S)$, $T_2(S)$, and $T_3(S)$. Pictures of the resulting fractals are given on pgs 246–8 of Peitgen, Jurgens & Saupe (1992a).

What sort of order can be brought to this table of pictures? Connectivity properties may be the most obvious: they allow one to classify fractals.

dusts (totally disconnected, Cantor sets),

dendrites (singly connected throughout, without loops),

multiply connected (connected with loops), and

hybrids (infinitely many components each containing a curve).

A parameter space map, painting points according to which of the four behaviors the corresponding fractal exhibits, did not reveal any illuminating patterns. However, sometimes (though not always—certainly not in the Cantor set cases, for example) in the unit square S there are finite collections of line segments that are preserved in $T_1(S) \cup T_2(S) \cup T_3(S)$. In

[1]Stop the presses: this conjecture has been proved in Lawler, Werner, & Schramm (2000).

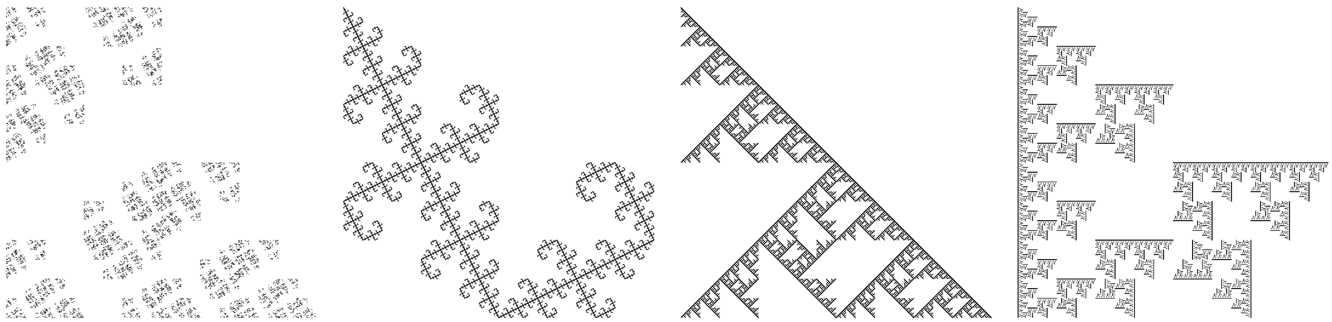

Figure 1: Relatives of the Sierpinski gasket: Cantor dust, dendrite, multiply connected, and hybrid. Can you find preserved line segments in the last three?

the cases where they could be found, these did give a transparent reason for the connectivity properties. This approach was generated by the student, looking for patterns by staring at the examples for hours on end.

What can we make of the observation that different collections of line segments work for different IFS? The student speculated that there is a *universal shape*, perhaps a union of some of the line segments from several examples, whose behavior under one application of T_1, T_2, and T_3 determines the connectivity form of the limiting fractal. This is an excellent question to be raised by a freshman, especially in a self-directed investigation.

This is just one example. Fractal geometry may be unique in providing such a wealth of visually motivated, but analytically expressed, problems. Truly, there is treasure everywhere.

4 Something new is always happening

New mathematics is coming up all the time; ours is a very lively field. However, many new developments are at an advanced level, often comprehensible only to experts having years of specialized training. To be sure, deep mathematical discoveries abound in fractal geometry, too. But because pictures are so central, here many advances have visual expressions that honestly reveal some of the underlying mathematics. New developments in retroviruses or in quantum gravity are unlikely to be comprehensible at anything other than a superficial level to general education students. They hear *about* the advances, but not *why* or *how* they work. The highly visual aspect of fractal geometry has allowed us to incorporate the most recent work into our courses in a serious way.

Here we describe one new development, and mention another to be explored in the next chapter.

4.1 Fractal lacunarity

It is difficult to imagine an introductory course on fractals that does not include computing dimensions of self-similar fractals. (See Chapters 5, 12, and 15, for example.) The calculations are straightforward, a skill mastered without excessive effort. Moreover, the idea generalizes to data from experiments, opening the way for a variety of student projects. However, one of the earliest exercises we assign points out a limitation of dimension: quite different-looking sets can have the same dimension. For example, all four fractals in Figure 1 have dimension $\log(3)/\log(2)$. The Sierpinski carpets of Figure 2 (Plate 318 of Mandelbrot (1982)) both subdivide the unit square into 49 pieces, each scaled by $\frac{1}{7}$, and delete nine of these pieces. So both have dimension $\log(40)/\log(7)$. On the left, these holes are distributed uniformly, on the right they are clustered together into one large hole in the middle. *Lacunarity* is one expression of this difference, and is another step in characterizing fractals through associated numbers. Here the number represents the distribution of holes or gaps, *lacunae*, in the fractal. This reinforces for students the relation between numbers and the visual aspects they are meant to represent. But also, this is current work, and even some of the basic issues are not yet settled. With this, our students see science as it is developing, and can understand some components of the debate.

To give an example of the kinds of results accessible to students having some familiarity with sequences and calculus, we describe an approach to the fractals of Figure 2. For a subset $A \subset \mathbf{R}^2$, the ϵ-thickening is defined as

$$A_\epsilon = \{\mathbf{x} \in \mathbf{R}^2 : d(\mathbf{x}, \mathbf{y}) \leq \epsilon \text{ for some } \mathbf{y} \in A\}$$

where $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$.

Now suppose A is either of the Sierpinski carpets in Figure 2. For large ϵ, A_ϵ fills all the holes of A and the area of A_ϵ, $|A_\epsilon|$, is $1 + 4\epsilon + \pi\epsilon^2$. As $\epsilon \to 0$, the holes of A become visible and increase the rate at which $|A_\epsilon|$ decreases. Calculations with Euclidean shapes—points, line segments, and circles, for example—show $|A_\epsilon| \approx L \cdot \epsilon^{2-d}$, where d is the dimension of the object. This relation can be used to compute the dimension, a technique developed by Minkowski and Bouligand. A first approach to lacunarity is the prefactor L, or more precisely, $1/L$, if the limit exists.

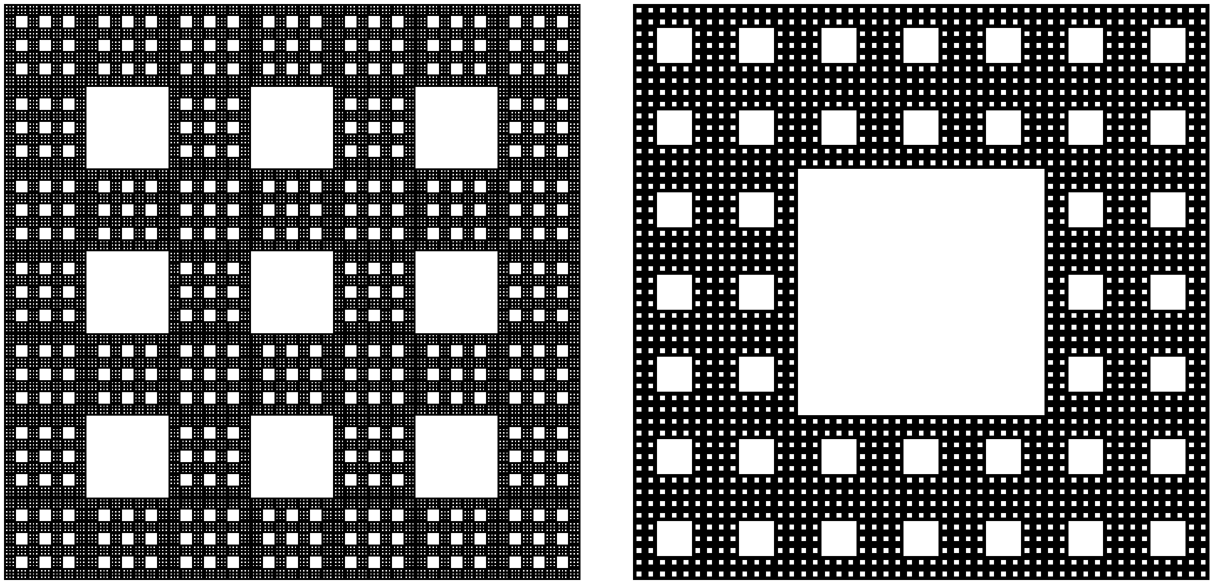

Figure 2: Two Sierpinski carpet fractals with the same dimension.

A general Sierpinski carpet is made with initiator the filled-in unit square, and generator the square with M squares of side length s removed. The iteration process next covers the complement of these M holes with N copies of the generator, each scaled by r. (Note the relation $1 - Ms^2 = Nr^2$.) For the carpet on the left side of Fig. 2 we see $M = 9$, $s = \frac{1}{7}$, $N = 40$, and $r = \frac{1}{7}$; on the right $M = 1$, $s = \frac{3}{7}$, $N = 40$, and $r = \frac{1}{7}$.

It is well known that for the box-counting dimension the limit as $\epsilon \to 0$ can be replaced by the sequential limit $\epsilon_n \to 0$, for ϵ_n satisfying mild conditions. Although the prefactor is generally more sensitive than the exponent, we begin with the sequence $\epsilon_n = sr^{n-1}/2$. For Sierpinski carpets A it is not difficult to see A_{ϵ_n} fills all holes of generation $\geq n$, while holes of generation $m < n$ remain. They are squares of side length $s(r^{m-1} - r^{n-1})$. Straightforward calculation gives

$$|A_{\epsilon_n}| = (4\epsilon_n + \pi\epsilon_n^2) + Ms^2\Big(\Big(\frac{2}{1-Nr}r^n - \frac{1}{1-N}r^{2n}\Big) + (Nr^2)^n\Big(\frac{1}{1-Nr^2} - \frac{2}{1-Nr} + \frac{1}{1-N}\Big)\Big).$$

Using $L \approx |A_{\epsilon_n}|\epsilon_n^{d-2}$, we obtain

$$L \approx M2^{2-d}s^d\Big(\frac{1}{1-Nr^2} - \frac{2}{1-Nr} + \frac{1}{1-N}\Big).$$

Substituting in the values of M, s, N, and r, we obtain $L \approx 1.41325$ and $L \approx 1.26026$ for the left and right carpets. So provisionally, the lacunarities are 0.707589 and 0.793487, agreeing with the notion that higher lacunarity corresponds to a more uneven distribution of holes.

Unfortunately, different sequences ϵ_n can give different values of L. Several approaches are possible, but one that is relatively easy to motivate and implement is to use a logarithmic average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{|A_{e^{-t}}|}{(2e^{-t})^{2-d}} dt.$$

The 2 in the denominator is a normalizing factor. For these carpets, this reduces to

$$\frac{Ms^d}{\log(1/r)}\Big(\frac{1}{1-Nr^2}\frac{1-r^{2-d}}{2-d} - \frac{2}{1-Nr}\frac{1-r^{1-d}}{1-d} + \frac{1}{1-N}\frac{1-r^{-d}}{-d}\Big).$$

Substituting in the values of M, s, N, and r, we obtain 1.305884 and 1.164514 for the left and right carpets. The respective lacunarities are 0.765765 and 0.858727.

These calculations involve simple geometry and can be extended easily to gaskets, their relatives, and the like. Even as the concepts continue to evolve, this is a rich source of ideas for student projects. Comparison with other lacunarity candidate measures—crosscut (Mandelbrot, Vespignani & Kaufman (1995)) and antipodal correlations (Mandelbrot & Stauffer (1994)), among others—in simple cases, is yet another source of projects. This has proven especially interesting because it shows students first-hand some of the issues involved in defining a measurement of a delicate property. Without being too heavy-handed, we point out in calculus that the definitions have been well-established for centuries. And even students in general education courses can appreciate the visual issues involved in the clustering of the *lacunae*.

4.2 Fractals in finance

As of this writing, the most common models of the stock market are based on Brownian motion. In fact, the first mathematical formulation of Brownian motion was Louis Bachelier's 1900 model of the Paris bond market. However, comparison with data instantly reveals many unrealistic features of Brownian motion $X(t)$. For example, $X(t_1) - X(t_2)$ and $X(t_3) - X(t_4)$ are independent for $t_1 < t_2 < t_3 < t_4$, and $X(t_1) - X(t_2)$ is Gaussian distributed with mean 0 and variance $|t_1 - t_2|$. That is, increments over disjoint time intervals are independent of one another, and the increments follow the familiar bell curve, so large increments are very rare. The latter is called the *short tails* property.

Are these reasonable features of real markets? Why should price changes one day be independent of price changes on a previous day? Moreover, computing the variance from market data assembled over a very long time, events of 10σ, for example, occur with enormously much higher frequency than the Gaussian value, which is (!) 10^{-24}. Practitioners circumvent these problems by a number of *ad hoc* fixes, adding up to a feeling similar to that produced by Ptolemy's cosmology: add enough epicycles and you can match any observed motion of the planets. Never mind the problems produced by the physicality of the epicycles, among other things. (Of course, in finance the situation is much worse. No one has a collection of epicycles that predicts market behavior with any reliability at all.)

In the 1960s, Mandelbrot proposed two alternatives to Brownian motion models. Mandelbrot (1963) had increments governed by the Lévy stable distribution (so with long tails), but still independent of one another. In 1965 Mandelbrot proposed a model based on fractional Brownian motion (See Mandelbrot (1997).) This model consequently had increments that are dependent, though still governed by the Gaussian distribution. Both are improvements, in different ways, of the Brownian motion models.

It is a considerable surprise, then, that Mandelbrot found a better model, and in addition a simple collection of *cartoons*, basically just iterates of a broken line segment, that by varying a single parameter can be tuned to produce graphs indistinguishable from real market data. The point, of course, is not to just make *Pick the Fake* quizzes that market experts fail, though to be sure, that has some entertainment and educational value. All these cartoons have built in the self-affinity observed in real data. Pursuing the goal of constructing the most parsimonious models accounting for observation, these cartoons suggest that dependence and non-Gaussian distributions may be a consequence of properly tuned self-affinity. More detail is given in the next chapter.

Finally, these cartoons are a perfect laboratory for student experimentation.

5 Conclusion

Some view science, perhaps especially mathematics, as a serious inquiry that should remain aloof from popular culture. Many of these people regret our teaching of fractal geometry, because its images have been embraced by popular culture.

We take the opposite view. As scientists, our social responsibility includes contributing to the scientific literacy of the general population. That fractal geometry has the visual appeal to excite wide interest is undeniable. This introduction argued that fractal geometry has the substance to engage non-science students in mathematics, in a serious way and to a greater degree than any other discipline of which we are aware. The chapters of this volume amplify this position by showing how a wide variety of teachers have done this in many settings.

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Chapter 2

Unsolved Problems and Still-Emerging Concepts in Fractal Geometry

Benoit B. Mandelbrot

The preceding chapter sketches a striking property of fractal geometry. Its first steps are, both literally and demonstrably, childishly easy. But high rewards are found just beyond those early steps. In particular, forbiddingly difficult research frontiers are so very close to the first steps as to be understood with only limited preparation. Evidence of this unique aspect of fractal geometry is known widely, but scattered among very diverse fields. It is good, therefore, to bring a few together. A fuller awareness of their existence is bound to influence many individuals' and institutions' perception of the methods, goals, and advancements of fractal geometry.

1 Introduction

"You find fractals easy? This is marvelous." Thus begins my response to an observation that is sometimes heard. "If you are a research mathematician, the community needs you to solve the challenging problems in this nice long list I carry around. If you are a research scientist, you could help to better analyze the important natural phenomena in this other long list."

The first half-answer is elaborated in Section 2. The point is that fractal geometry has naturally led to a number of compelling mathematical conjectures. Some took 5, 10, or 20 years to prove, others—despite the investment of enormous efforts—remain open and notorious. If anything, what slows down the growth of fractal-based mathematics is the sheer difficulty of some of its more attractive and natural portions.

The second-half answer is elaborated in Section 3. The point is that, among other features, fractal geometry is, so far, the only available language for the study of roughness, a concept that is basic and related to our senses, but has been the last to give rise to a science. In many diverse pre-scientific fields, the absence of a suitable language delays the moment when some basic problems could be attacked scientifically. In other instances, it even delays the moment when those problems could be stated.

2 From simple visual observation to forbiddingly difficult mathematical conjecture

A resolutely purist extreme view of art holds that great achievements must be judged for themselves, irrespective of their period and the temporary failures that preceded their being perfected. In contrast, the most popular view attaches great weight to cultural context and mutual influences, and more generally tightly links the process and its end-products. Some works do not survive as being excellent but as being representative or historically important. For example, a resolutely sociological extreme view that we do not share holds that Beethoven's greatness in his time and ours distracts from the more important appreciation of his contemporaries. Few persons, and not even all teachers, are aware that a very similar conflict of views exists in mathematics.

As widely advertised, the key product of mathematics consists in theorems; in each, assumptions and conclusions are linked by a proof. It is also well known that many theorems began in the incomplete status of conjectures that include assumptions and conclusions but lack a proof. The iconic example was a conjecture in number theory due to Fermat. After a record-breaking long time it led to a theorem by Wiles. Conjectures that resist repeated attempts at a proof acquire an important role, in fact, a very peculiar one. On occasion the news that an actual proof has made a conjecture into a theorem is perceived as a letdown, while it is suggested that these conjectures' main value resides in the insights provided by both the unsuccessful and the successful searches for a proof.

Be that as it may, fractal geometry is rich in open conjectures that are easy to understand, yet represent deep mathematics. First, they did not arise from earlier mathematics, but in the course of practical investigations into diverse natural sciences, some of them old and well established, others newly revived, and a few altogether new. Second, they originate in careful inspections of actual pictures generated by computers. Third, they involve in essential fashion the century-old mathematical *monster shapes* that were for a long time guaranteed to lack any contact with the real world. Those fractal conjectures attracted very wide attention in the professions but elude proof. We feel very strongly that those fractal conjectures should not be reserved for the specialists, but should be presented to the class whenever possible. The earlier, the better. To dispel the notion that all of mathematics was done centuries ago, nothing beats being able to understand appropriate problems no one knows how to solve. Not all famous unsolved problems will work here: the Poincaré conjecture cannot be explained to high school students in an hour or a few. But many open fractal conjectures can.

For the reasons listed above, the questions raised in this chapter bear on an issue of great consequence. Does pure (or purified) mathematics exist as an autonomous discipline, one that can and should develop in total isolation from sensations and the material world? Or, to the contrary, is the existence of totally pure mathematics a myth?

The role of visual and tactile sensations. The ideal of pure mathematics is associated with the great Greek philosopher Plato (427?–347 BC). This (at best) mediocre mathematician used his great influence to free mathematics from the pernicious effects of the real world and of sensations. This position was contradicted by Archimedes (287–212 BC), whose realism I try to emulate.

Indeed, my work is unabashedly dominated by awareness of the importance of the messages of our senses. Fractal geometry is best identified in the study of the notion of roughness. More specifically, it allows a place of honor to full-fledged pictures that are as detailed as possible and go well beyond mere sketches and diagrams. Their original goal was modest: to gain acceptance for ideas and theories that were developed without pictures but were slow to be accepted because of cultural gaps between fields of science and mathematics. But those pictures then went on to help me and many others generate new ideas and theories. Many of these pictures strike everyone as being of exceptional and totally unexpected beauty. Some have the beauty of the mountains and clouds they are meant to represent; others are abstract and seem wild and unexpected at first, but after brief inspection appear totally familiar. In front of our eyes, the visual geometric intuition built on the practice of Euclid and of calculus is being retrained with the help of new technology.

Pondering these pictures proves central to a different philosophical issue. Does the beauty of these mathematical pictures relate to the beauty that a mathematician rooted in the twentieth century mainstream sees in his trade after long and strenuous practice?

2.1 Brownian clusters: fractal islands

The first example, introduced in Mandelbrot (1982), is a wrinkle on Brownian motion. The historical origins of random walk (drunkard's progress) and Brownian motion are known and easy to understand, at least qualitatively. From this, it is simple to motivate the definition of the Wiener Brownian motion: a random process $B(t)$ with increments $B(t+h) - B(t)$ that obey the Gaussian distribution of mean 0 and variance h, and that are independent over disjoint intervals.

For a given time L, the *Brownian bridge* $B_{\text{bridge}}(t)$ is defined by

$$B_{\text{bridge}}(t) = B(t) - (t/L)B(L),$$

for $0 \leq t \leq L$. Taking $B(0) = 0$, we find $B_{\text{bridge}}(L) = B_{\text{bridge}}(0)$. Combining one Brownian bridge in the x-direction and one in the y-direction and erasing time yields a *Brownian plane cluster* Q. Because we use Brownian bridges to construct it, the Brownian plane cluster is a closed curve. See Figure 1. An example of a well-known and fully proven

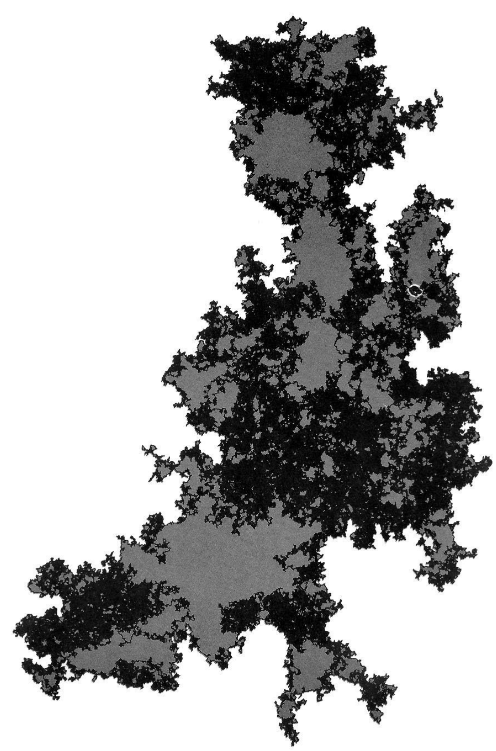


Figure 1: A Brownian plane cluster; Plate 243 of Mandelbrot (1982).

fact is that the fractal dimension of Q is $D = 2$. This result is important but not really perspicuous, because the big holes seem to contradict the association of $D = 2$ with plane-filling curves. Results that are well known and not perspicuous are not for the beginner.

Let us proceed to the *self-avoiding planar Brownian motion* $\tilde{Q}$. It is defined in Mandelbrot (1982) as the set of points of the cluster Q accessible from infinity by a path that fails to intersect Q. That is, $\tilde{Q}$ is the *hull* of Q, also called its *boundary* or *outer edge*. The hull $\tilde{Q}$ is easy to comprehend because it lacks double points. The unanswered question associated with it is the **4/3 Conjecture**, that $\tilde{Q}$ has fractal dimension $4/3$.

An early example of Q, and hence of $\tilde{Q}$ is seen in Figure 1. It looks like an island with an especially wiggly coastline, and experience suggested its dimension is approximately $4/3$. This comparison with islands made the 4/3 conjecture sensible and plausible in 1982 and it remains sensible and plausible to students; that it remained a conjecture for many years is something they can appreciate. Numerical tests and physicists' heuristics were added to the empirical evidence and the conjecture was proved in Lawler, Werner, & Schramm (2000).

2.2 The Mandelbrot set

Second example: In the past, music could be both popular and learned, but *elitists* believe that this is impossible today. For mathematics, the issue was not raised because no part of it could be called a part of popular culture. Providing a counterexample, no other modern mathematical object has become part of both scientific and popular culture as rapidly and thoroughly as the Mandelbrot set. Moreover, an algorithm for generating this set is readily mastered by anyone familiar with elementary algebra. Thousands of people, from middle school children to senior researchers and Fields Medalists, have written programs to visualize various aspects of the Mandelbrot set.

Recall the simplest algorithm: a complex number c belongs to the Mandelbrot set M if and only if the sequence $z_0, z_1, z_2, \ldots$ stays bounded, where $z_0 = 0$ and $z_{i+1} = z_i^2 + c$.

For instance, the sequence can stay bounded by converging to a fixed point or to a cycle. Denote by M_0 the set of all c for which this is true. Of course, $M_0 \subset M$. In fact, M_0 is of interest to the students of dynamics, hence my original investigations were of M_0, not of M. Interest shifted to M because producing pictures of M is easy. By contrast, to test if $c \in M_0$, we first generate several hundred or thousand points of the sequence $z_0, z_1, z_2, \ldots$, and test if for large enough i there is an n for which $|z_{i+n} - z_i|$ is very small. This suggests convergence to a cycle of length n. (An impractical theoretical alternative is to solve the 2^n-degree polynomial equation $f_c^n(z) = z$, where $f_c(z) = z^2 + c$, then test the stability of the n-cycle by a derivative condition:

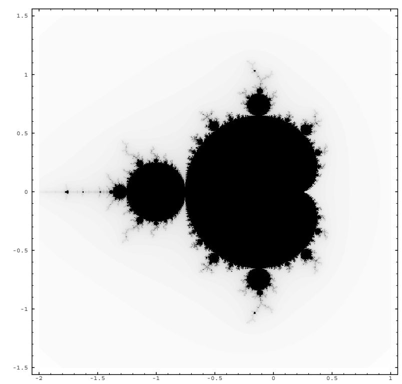


Figure 2: The Mandelbrot set.

$1 > |f_c'(w_1) \cdots f_c'(w_n)| = 2^n \cdot |w_1 \cdots w_n|$. Here the points $w_1, \ldots, w_n$ of the n-cycles are the sequences of successive z_i for different z_0. In general, for each c there are several n-cycles, but at most one is stable.)

Computer approximations of M_0 actually yield a set smaller than M_0, and computer approximations of M actually yield a set larger than M. Extending the duration of the computation seemed to make the two representations converge to each other and to an increasingly elaborate common limit. Furthermore, when c is an interior point of M, not too close to the boundary, it was easily checked that a finite limit cycle exists: the steps outlined above converge fairly rapidly for c not too close to the boundary. Those observations led me to conjecture that M is identical to M_0 together with its limits points, that is, $M = cl(M_0)$, the closure of M_0.

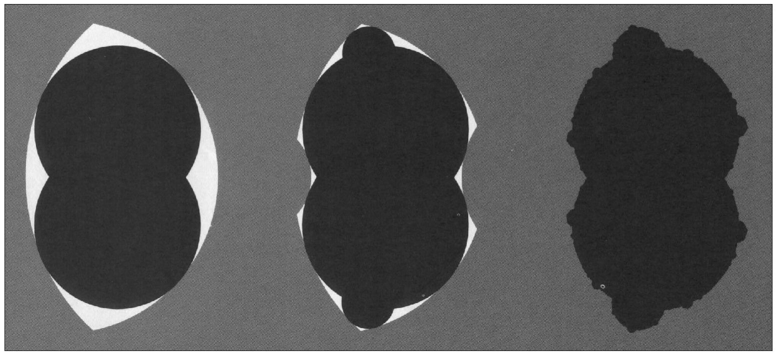
In terms of its being simple and understandable without any special preparation, this conjecture is difficult to top. But after almost twenty years of study, it remains a conjecture. With the proof of Fermat's last theorem, the conjecture $M = cl(M_0)$ may have been promoted to illustrating the shortest distance between a simple idea (in this case, complete with popular pictures) and deep, unsolved mathematics. (Not so simple is the usual restatement of this conjecture: that M is locally connected.)

2.3 Dimensions of self-affine sets

The first tool for quantifying self-similar fractals is dimension. For a fractal consisting of N pieces, each scaled in all directions by a factor of r, the dimension D is given by

$$D = \frac{\log(N)}{\log\left(\frac{1}{r}\right)}.$$

This is easy to motivate, trivial to compute. Working through several examples, students soon develop intuition for the visual signatures of low- and high-dimensional fractals. The generalization to self-similar fractals having different scal-



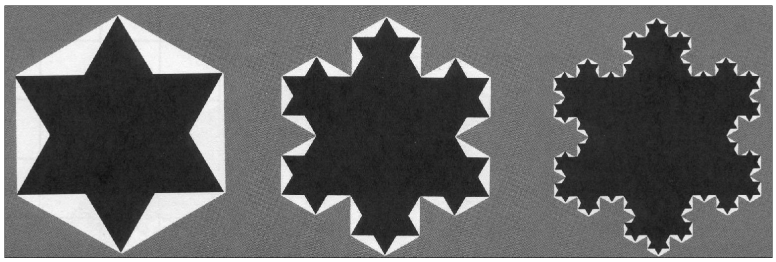


Figure 3: Top: Osculating circles outlining a Jordan curve limit set from inside and outside, Plate 177 of Mandelbrot (1982). Bottom: Osculating triangles outlining the Koch snowflake curve from inside and outisde, Plate 43 of Mandelbrot (1982).

ings for different pieces is not difficult. For a fractal consisting of N pieces, the i^{th} piece scaled by a factor of r_i, the dimension D is the unique solution of the *Moran equation*

$$\sum_{i=1}^{N} r_i^D = 1.$$

Often this must be solved numerically, but this is not a difficulty given today's graphing calculators and computer algebra packages.

The simplicity of these calculations leads some people to believe that calculating dimensions is a simple process. This is a misperception resulting from the almost exclusive reliance on self-similar fractals for examples. The case of self-affine fractals, where the pieces are scaled by different factors in different directions, is much more difficult. Although some special cases are known, no simple variant of the Moran equation has been found. Kenneth Falconer describes the situation this way, "Obtaining a dimension formula for general self-affine sets is an intractable problem." (Falconer (1990), 129.) By simply changing the scaling factors in one direction, a completely straightforward exercise becomes tremendously difficult, perhaps without general solution.

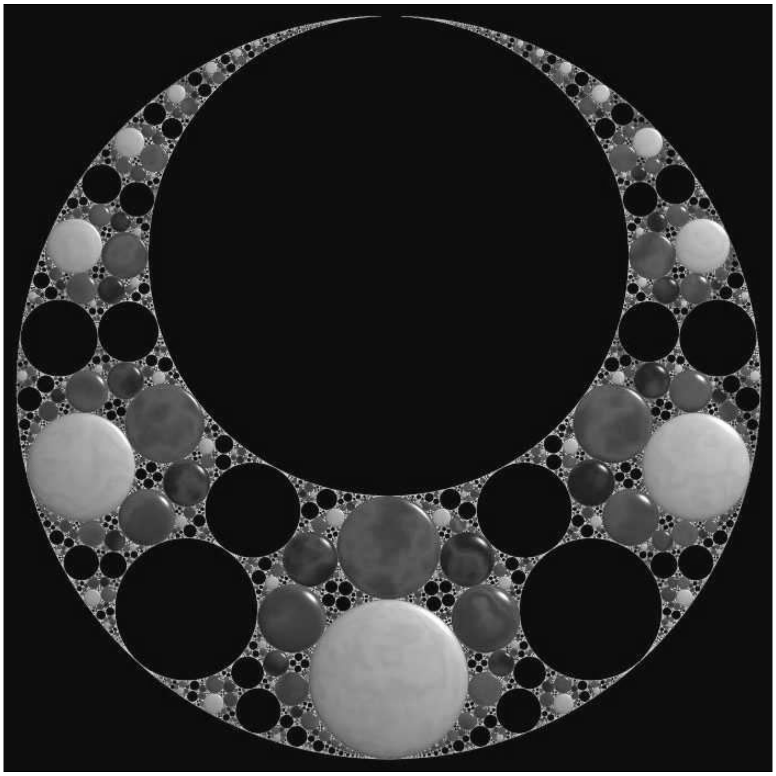
2.4 Limit sets of Kleinian groups

A collection of Möbius transformations of the form $z \to (az + b)/(cz + d)$ defines a group that Poincaré called Kleinian. With few exceptions, their limit sets S are fractal. For the closely related groups based on geometric inversions in a collection $C_1, C_2, \ldots, C_n$ of circles, there is a well-known algorithm that yields S in the limit. But it converges with excruciating slowness as seen in Plate 173 of Mandelbrot (1982). For a century, the challenge to obtain a fast algorithm remained unanswered, but it was met in many cases in Chapter 18 of Mandelbrot (1982). See also Mandelbrot (1983). In the case of this construction, fractal geometry did not open a new mathematical problem, but helped close *a very old* one.

In the new algorithm, the limit set of the group of transformations generated by inversions is specified by covering the complement of S by a denumerable collection of circles that *osculate* S. The circles' radii decrease rapidly, therefore their union outlines S very efficiently.

When S is a Jordan curve (as on Plate 177 of Mandelbrot (1982)), two collections of osculating circles outline S, respectively from the inside and the outside. They are closely reminiscent of the collection of osculating triangles that outline Koch's snowflake curve from both sides (Figure 3). Because of this analogy, the osculating construction seems, after the fact, to be very natural. But the hundred year gap before it was discovered shows it was not obvious. It came only after respectful examination of pictures of many special examples.

A particularly striking example is seen in Figure 4, called "Pharaoh's breastplate," Ken Monks' improved rendering of Plate 199 of Mandelbrot (1982). A more elaborate version of this picture appears on the cover of Mandelbrot (1999). This is the limit set of a group generated by inversion in the six



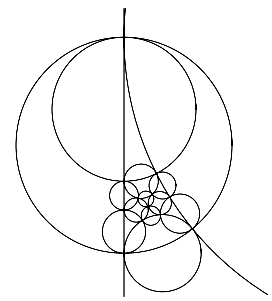


Figure 4: Left: Pharaoh’s breastplate. See the color plates. Right: the six circles generating Pharaoh’s breastplate, together with a few circles of the breastplate for reference.

circles drawn as thin lines on the small accompanying diagram. Here, the basic osculating circles actually belong to the limit set and do not intersect (each is the limit set of a Fuchsian subgroup based on three circles). The other osculating circles follow by all sequences of inversions in the six generators, meaning that each osculator generates a *clan* with its own *tartan* color.

By inspection, it is easy to see that this group also has three additional Fuchsian subgroups, each made of four generators and contributing full circles to the limit set.

Pictures such as Figure 4 are not only aesthetically pleasing, but they breathe new life into the study of Kleinian groups. Thurston’s work on hyperbolic geometry and 3-manifolds opens up the possibility for limit sets of Kleinian group actions to play a role in the attempts to classify 3-manifolds. The Hausdorff dimension of these limit sets has been studied for some time by Bishop, Canary, Jones, Sullivan, Tukia, and others. The group G that generates the limit set gives rise to another invariant, the *Poincaré exponent*

$$\delta(G) = \inf \left\{ s : \sum_{g \in G} \exp\left(-s\rho\left(0, g(0)\right)\right) \right\} < \infty$$

where ρ is the hyperbolic metric. Under fairly general conditions, the Poincaré exponent of a Kleinian group equals the Hausdorff dimension of the limit set of the group. See Bishop & Jones (1997), for example.

This is an active area of research: much remains to be done.

3 “Mathematics is a language”: the emergence of new concepts

History tells us that the great Josiah Willard Gibbs (1839–1903) made this remark at a Yale College Faculty meeting devoted to the reform of foreign language requirements (some faculty issues never die!). The context may seem undignified or amusing, but, in fact, Gibbs’s words bring forth a deep issue. To express subtle scientific ideas, one often needs new *words* that are subtler than those of ordinary language.

As background, everyone knows that some great books deservedly became classics because they provided, for the first time, a new language in which personal emotions—that the reader would feel but not be able to express—could be both refined and made public. This is not at all a matter of coining new words for old concepts but of making altogether new concepts emerge.

Advances in the sciences are assessed in diverse ways, one of which is the emergence of new scientific concepts. Indeed, the facile precept that the first step is to observe then measure, sounds less compelling when the object of study is an undescribable mess and all the measurements that readily come to mind disagree or even seem self-contradictory. This is why the point of passage from prescientific to scientific investigation is often marked by what Thomas Kuhn called *change of paradigm*. Sometimes this includes the appearance of a suitable new language, without which observations could not be made and quantified.

3.1 Fractals are a suitable language for the study of roughness wherever it is encountered

Let us ponder the ubiquity of the notion of roughness and its lateness in becoming formalized. Many sciences arose directly from the desire to describe and understand some basic messages the brain receives from the senses. Visual signals led to the notions of bulk and shape and of brightness and color. The sense of heavy versus light led to mechanics and the sense of hot versus cold led to the theory of heat. Other signals (for example, auditory) require no comment. Proper measures of mass and size go back to prehistory and temperature, a proper measure of hotness, dates to Galileo.

Against this background, the sense of smooth versus rough suffered from a level of neglect that is noteworthy though hardly ever pointed out. Not only does the theory of heat have no parallel in a theory of roughness, but temperature itself had no parallel until the advent of fractal geometry. For example, in the context of metal fractures, roughness was widely measured by a root mean square deviation from an interpolating plane. In other words, metallurgists used the same tool as finance experts used to measure volatility. But this measurement is inconsistent. Indeed, different regions of a presumably homogeneous fracture emerged as being of different r.m.s. volatility. The same was the case for different samples that were carefully prepared and later broken following precisely identical protocols.

To the contrary, as shown in Mandelbrot, Passoja & Paullay (1984) and confirmed by every later study, the fractal dimension D, a characteristic of fractals, provides the desired invariant measure of roughness. The quantity $3 - D$ is called the codimension or Hölder exponent by mathematicians and now called the roughness exponent by metallurgists.

The role played by exponents must be sketched here. It is best in this chapter to study surfaces through their intersections by approximating orthogonal planes. Had these functions been differentiable, they could be studied through the derivative defined by $P'(t) = \lim_{\epsilon\to 0}(1/\epsilon)[P(t+\epsilon) - P(t)]$. For fractal functions, however, this limit does not exist and the local behavior is, instead, studied through the parameters of a relation of the form $dP \sim F(t)(dt)^{\alpha}$. Here $F(t)$ is called the prefactor, but the most important parameter is the exponent $\alpha = \lim_{\epsilon\to 0}\{\log[P(t+\epsilon) - P(t)]/\log\epsilon\}$.

There is an adage that, when you own only a hammer, everything begins to look like a nail. This adage does not apply to roughness.

3.2 Fractals and multifractals in finance

Versions of the Brownian motion model mentioned in Section 2.1 are widely used to model aspects of financial markets. In fact, and contrary to common belief, the first analysis of Brownian motion was not advanced in 1905 by Einstein. In 1900 Bachelier had already developed Brownian motion to study the stock market.

Despite this historical precedent, successive differences of real data sampled at equal time intervals reveal even on cursory investigation that Brownian models are very far from being tolerable. Most visibly, (1) the width of the *central band* is not constant, but varies substantially, (2) the excursions from the central band are so large as to be astronomically unlikely in the Brownian case, and (3) the excursions are not independent, but occur in clumps, often when the underlying band is widest. Figure 5 illustrates these differences.

Ad hoc fixes can account for each of these failures of the Brownian model, but very rapidly become far too complicated for anybody, especially for courses not addressed to experts. The fractal/multifractal approach of Mandelbrot (1997) is much more elegant. It provides a unified way to synthesize all, and moreover introduces a family of parameterized cartoon models suitable for student exploration.

Let us dwell on what is happening. Compared with well-developed standard mathematical finance, the fractal cartoons are incomparably more satisfactory. But they are far simpler than the first stages of standard finance, so simple that they have been immediately incorporated into both *Fractal Geometry for Non-Science Students* (a course primarily for humanities students) and *Fractal Geometry: Techniques and Applications* (a course for sophomore-junior math and science students) at Yale. In effect, students are invited to participate in discussions between experts. They are amazed by the realistic appearance of forgeries made with these cartoons. Showing the class a collection of real data and forgeries always produces interesting results. Students disagree, sometimes with great animation, about which are real and which are forgeries. The *inverse problem*, finding a cartoon to create a forgery of a particular data set, has been a source of interesting student projects, some quite creative. After studying background in the different visual signatures of long tails and global dependence, students are amazed at how slight changes in the cartoon generator can achieve both effects.

The basic construction of the cartoon involves an *initiator* and a *generator*. The process to be iterated consists of replacing each copy of the initiator with an appropriately rescaled copy of the generator. For a first cartoon, the initiator is the diagonal of the unit square, and the generator is the broken line with vertices $(0, 0)$, $(4/9, 2/3)$, $(5/9, 1/3)$, and $(1, 1)$. Fig-

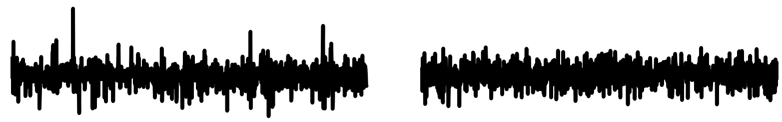


Figure 5: Left: differences in successive daily closing prices for four years of EMC data. Right: successive differences of the same number of steps in one-dimensional Brownian motion.

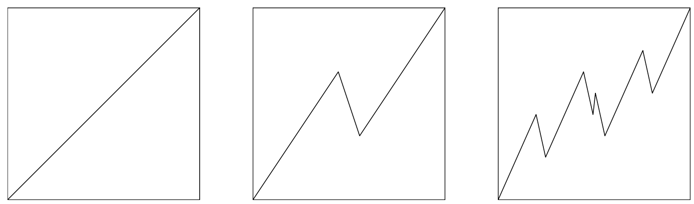


Figure 6: The initiator, generator, and first iterate of a non-random Brownian motion cartoon.

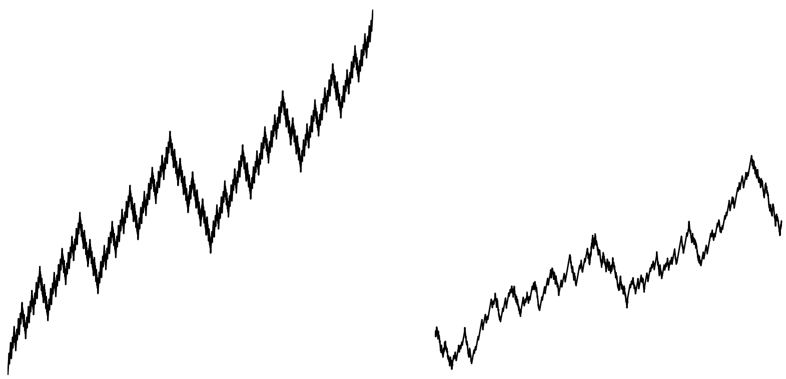


Figure 7: Left: the 6th iterate of a non-random Brownian cartoon. Right: the 6th iterate of a randomized Brownian cartoon.

ure 6 shows the initiator (left), generator (middle), and first iteration of the process (right).

To get an appreciation for how quickly the jaggedness of these cartoons grows, the left side of Figure 7 shows the 6th iterate of the process.

Self-affinity is guaranteed because it is built into the process; each piece is an appropriately scaled version of the whole. In this case, the scaling ratios have been selected to satisfy the square root condition of Brownian motion. The horizontal axis denotes time t, the vertical denotes price x. The first and third generator segments have $\Delta t_1 = \Delta t_3 = \frac{4}{9}$ and $\Delta x_1 = \Delta x_3 = \frac{2}{3}$; the middle segment has $\Delta t_2 = \frac{1}{9}$ and $\Delta x_2 = -\frac{1}{3}$. So for each generator segment we have $|\Delta x_i| = (\Delta t_i)^{1/2}$.

A cartoon is *unifractal* if there is a constant H so that for each generator segment $|\Delta x_i| = (\Delta t_i)^H$. If different H are needed for different segments, the cartoon is *multifractal*.

The left side of Figure 7 is far too regular to mimic any real data. But it can be randomized easily by shuffling the order in which the three pieces of the generator are put into each scaled copy. The right side of Figure 7 shows the result of this shuffling, for the sixth stage of the construction.

Instead of the graph itself, it is less common but far better to look at the increments. The cartoon sequence we have produced has jumps at uneven intervals: some at multiples of $1/3^n$, some at multiples of $1/9^n$. Because we rarely have detailed knowledge of the underlying dynamics generating real data, measurements usually are taken at equal time steps. To construct a sequence of appropriate increments, we sample the graph at fixed time intervals and subtract successive values obtained. Operationally, first make a list of time values for the sampling, then find the cartoon time values between which each sample value lies, and linearly interpolate between the cartoon values to find the sample value at the sample time.

Figure 8 illustrates how the statistical properties of the differences can be modified by making a simple adjustment in the generator. Fixing the points $(0, 0)$ and $(1, 1)$, we keep the middle turning points symmetrical: $(a, \frac{2}{3})$ and $(1 - a, \frac{1}{3})$, where a lies in the range $0 < a \leq \frac{1}{2}$. All pictures were constructed from the tenth generation, hence consist of $3^{10} =$

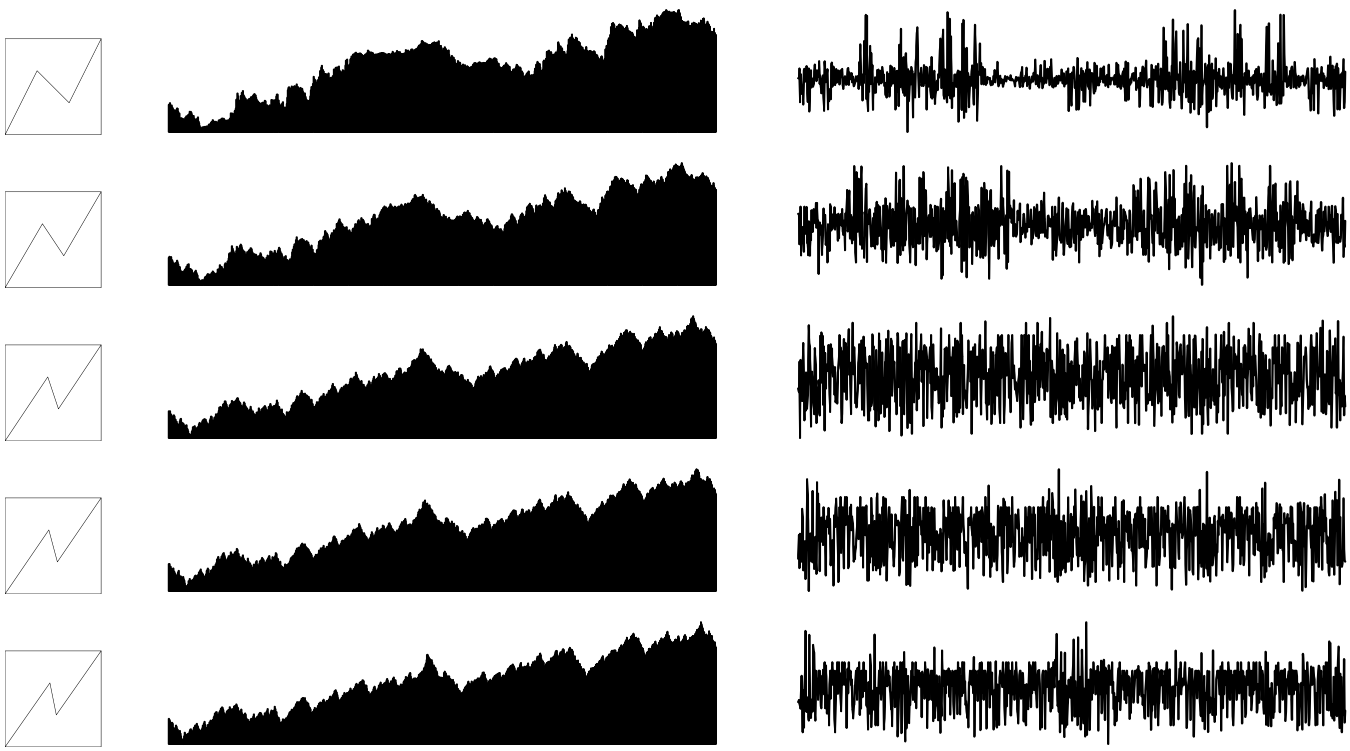


Figure 8: Generators, cartoons and difference graphs for symmetric cartoons with turning points $(a, \frac{2}{3})$ and $(1 - a, \frac{1}{3})$, for $a = 0.333$, 0.389, 0.444, 0.456, and 0.467. The same random number seed is used in all graphs.

59,049 intervals. The difference graphs are constructed by sampling at 1000 equal time steps.

Certainly, correlations are introduced as the point $(4/9, 2/3)$ is moved to the left. Of course, this is just the beginning. More detailed study reveals relations between the Hölder exponents and the slopes of the generator intervals, and properties of the multifractal measure can be extracted from the cartoons (Mandelbrot (1997)). The H-exponents and the $f(\alpha)$ curve are much too technical for *Fractal Geometry for Non-Science Students*, but are appropriate topics for the more mathematically sophisticated *Fractal Geometry: Techniques and Applications*. Even for this audience, these are challenging concepts. Yet these simple cartoons provide accessible introductions to some of the subtle mathematics of multifractals.

As a last example, we mention a fascinating theorem and a visual representation of its meaning. The Yale students taking *Fractal Geometry for Non-Science Students* in autumn of 1998 followed the development of Figure 9 with passion and helped improve it. The generator increments Δt represent *clock time*. Viewed in clock time, prices sometimes remain quiescent for long periods, and sometimes change with startling rapidity, perhaps even discontinuously. For these cartoons, clock time can be recalibrated to uniformize these changes in price variation. Basically, slow the clock during periods of rapid activity and speed it during periods of low activity. Students found the VCR a useful analog. Fast-forward through the commercials (low activity) and use slow-motion through the interesting bits (rapid activity).

For the cartoon generators, this is achieved by first finding the unique solution D of $|\Delta x_1|^D + \cdots + |\Delta x_n|^D = 1$, then defining the *trading time* generators by $\Delta T_i = |\Delta x_i|^D$. By changing to trading time, every multifractal price cartoon can be converted into a unifractal cartoon in multifractal time. Global dependence and long tails are unpacked in different ways by converting to trading time. Specifically, global dependence remains in the price vs. trading time record, but the long tails are absorbed into the multifractal nature of trading time.

Figure 9 shows a three-dimensional representation of this conversion. Note how the clock time-trading time curve compresses the flat regions and expands the steep regions of the price-clock time graph. Thus the long tails of the price-clock time graph are absorbed into the multifractal time measure. In addition, the dependence of increments is uniformized to fractional Brownian motion in the price-trading time graph. That is, the conversion to trading time decomposes long tails and dependent increments into different aspects of the graph.

Starting from a rough idea of such a representation, this picture evolved over about a week, through discussions with the class. Few things have excited the class as much as being

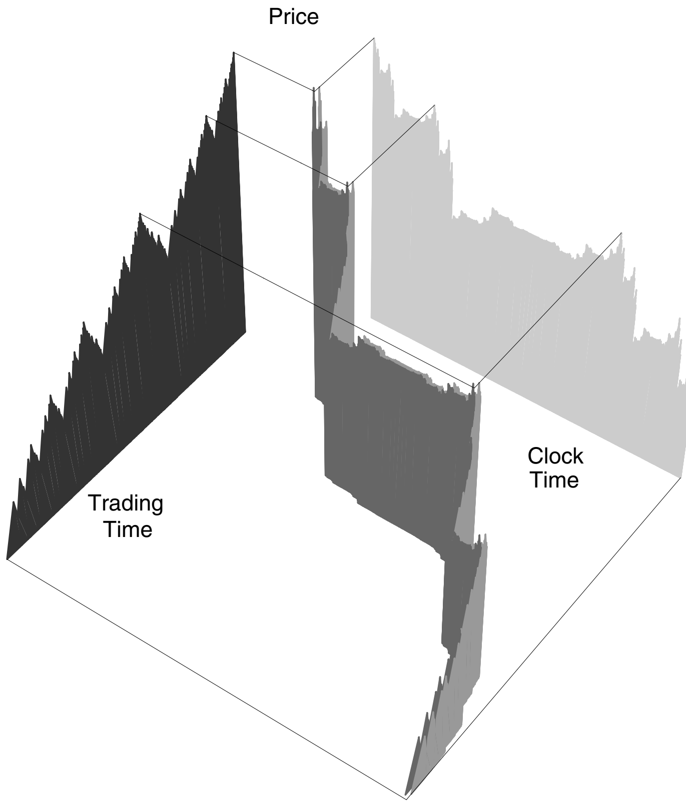


Figure 9: Converting the price-clock time graph to the price-trading time graph by means of the clock time-trading time graph.

involved, as a group, in the production of a figure to explain current research in the field.

4 Conclusion

A famous tongue-twister and test in Greek and evolution, due to E. H. Haeckel, proclaims that "ontogeny recapitulates phylogeny." In plainer English, the early growth of an individual repeats the evolution of its (his, her) ancestors. As argued elsewhere in this book (Chapter 3), this used to be the **BIG PICTURE** historical justification of *old math*—not a well-documented one. But teachers ought to welcome any well-documented **small picture** version that happens to come their way. Fractals deserve to be welcomed.

M.L. Frame & B.B. Mandelbrot
Fractals, Graphics, & Mathematics Education
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Chapter 3

Fractals, Graphics, and Mathematics Education[1]

Benoit B. Mandelbrot

1 Introduction

The fundamental importance of education has always been very clear to me and it has been very frustrating, and certainly not a good thing in itself, that the bulk of my working life went without the pleasures and the agonies of teaching. On the other hand, there is every evidence, in my case, that being sheltered from academic life has often been a necessary condition for the success of my research. An incidental consequence is that some of the external circumstances that dominated my life may matter to the story to be told here, and it will be good to mention them, in due time.

But past frustrations are the last thing to dwell upon in this book. Watching some ideas of mine straddle the chasm between the research frontier and the schools overwhelms me with a feeling of deep accomplishment. Clearly, for better or worse, I have ceased to be alone in an observation, a belief, and a hope, that keep being reinforced over the years.

The observation is that fractals—together with chaos, easy graphics, and the computer—enchant many young people and make them excited about learning mathematics and physics. In part, this is because an element of instant gratification happens to be strongly present in this piece of mathematics called fractal geometry. The belief is that this excitement can help make these subjects easier to teach to teenagers and to beginning college students. This is true even of those students who do not feel they will need mathematics and physics in their professions. This belief leads to a hope—perhaps megalomaniac—concerning the abyss which has lately separated the scientific and liberal cultures. It is a cliché, but one confirmed by my experience, that scientists tend to know more of music, art, history, and literature, than humanists know of any science. A related fact is that far more scientists take courses in the humanities than the other way around. So let me give voice to a strongly held feeling. An element of instant gratification happens to be strongly present in this piece of mathematics called fractal geometry. Would it be extravagant to hope that it could help broaden the small band of those who see mathematics as essential to every educated citizen, and therefore as having its place among the liberal arts?

The lost unity of liberal knowledge is not just something that old folks gather to complain about; it has very real social consequences. The fact that science is understood by few people other than the scientists themselves has created a terrible situation. One aspect is a tension between conflict of interest and stark ignorance: that vital decisions about science and technology policy are all too often taken either by people so closely concerned that they have strong vested interests, or by people who went through the schools with no math or science. Thus, every country would be far better off if understanding and appreciation for some significant aspect of science could become more widespread among its citizens. This demands a liberal education that includes substantial instruction in math.

Fractals prove to have many uses in technical areas of mathematics and science. However, this will not matter in this chapter. Besides, if fractals' usefulness in teaching is confirmed and proves lasting, this is likely to dwarf all their other uses.

This chapter shall assume all of you to have a rudimentary awareness or knowledge of fractals, or will one day become motivated to acquire this knowledge elsewhere. My offering is my book, Mandelbrot (1982), but there are many more

[1] Adapted from a closing invited address delivered at the Seventh International Congress of Mathematics Education (ICME-7), held in 1992 at Laval University of Quebec City. The text remains self-contained and preserves some of the original flavor; it repeats some points that were already used elsewhere in this book but bear emphasis.

sources at this point. For example, the website

`http://classes.yale.edu/math190a/Fractals/Welcome.html`

is a self-contained short course on basic fractal geometry.

I shall take up diverse aspects of a basic and very concrete question about mathematics education: what should be the relations—if any—between (a) the overall development of mathematics in history, (b) the present status of the best and brightest in mathematics research, and (c) the most effective ways of teaching the basics of the field?

2 Three mutually antagonistic approaches to education

By simplifying (strongly but not destructively), one can distinguish three mutually antagonistic approaches to mathematical education. The first two are built on *a priori* doctrine: the **old math**, dominated by (a) above, and the **new math**, dominated by (b). (I shall also mention a transitional approach between old and new math.) To the contrary, the approach I welcome would be resolutely pragmatic. It would encourage educational philosophy to seek points of easiest entry. In this quest, the questions of how mathematics research began and of its present state, are totally irrelevant.

To elaborate by a simile loaded in my favor, think of the task of luring convinced nomads into hard shelter. One could tempt them into the kinds of shelters that have been built long ago, in countries that happened to provide a convenient starting point in the form of caves. One could also try to tempt them into the best possible shelters, those being built far away, in highly advanced countries where architecture is dominated by structurally pure skyscrapers. But both strategies would be most ill-inspired. It is clearly far better to tempt our nomads by something that interests them spontaneously. But such happens precisely to be the case with fractals, chaos, easy graphics, and the computer. Hence, if their effectiveness becomes confirmed, a working pragmatic approach to mathematics education may actually be at hand. We may no longer be limited to the old and new math. Let me dwell on them for a moment.

2.1 The old math approach to mathematics education

The old math approach to mathematics education saw the teacher's task as that of following history. The goal was to guide the child or young person of today along a simplified sequence of landmarks in the progress of science throughout the history of humanity. An extreme form of this approach prevailed until mid-nineteenth century in Great Britain, the sole acceptable textbook of geometry being a translation of Euclid's *Elements*.

The folk-psychology behind this approach asserted with a straight face that the mental evolution of mankind was the product of historical necessity and that the evolution of an individual must follow the same sequence. In particular, the acquisition of concepts by the small child must follow the same sequence as the acquisition of concepts by humankind. Piaget taught me that such is indeed the case for concepts that children must have acquired before they start studying mathematics.

All this sounds like a version of "ontogeny recapitulates phylogeny," but it is safe to say that people had started developing mathematics well before Euclid. As a matter of fact, those who edited the *Elements* were somewhat casual and left a number of propositions in the form of an archaeological site where the latest strata do not completely hide some tantalizing early ones. To be brief, what we know of the origin of mathematics is too thin and uncertain to help the teacher.

Be that as it may, an acknowledged failing of old math was that the teacher could not conceivably move fast enough to reach modern topics. For example, the school mathematics and science taken up between ages 10 and 20 used to be largely restricted to topics humanity discovered in antiquity. As might be expected, teachers of old lit heard the same criticism. A curriculum once reserved to Masters of Antiquity was gradually changed to leave room for the likes of Shakespeare and of increasingly modern authors; in the USA, it had to yield room to American Masters, then to multicultural programs.

2.2 Transitional approaches to mathematics education

Concerns about old math are an old story. Consider two examples. In Great Britain, unhappiness with Euclid's *Elements* as a textbook fueled the reforms movement that led in 1871 to the foundation of the *Association for the Improvement of Geometrical Teaching* (in 1897 it was renamed the *Mathematical Association*). As a student in France around 1940, I heard about a reform movement that had flourished before 1900. It motivated Jacques Hadamard (1865–1963), a truly great man, to help high school instruction by writing Hadamard (1898), a modern textbook of geometry that stressed the notion of transformation. I was given a copy and greatly enjoyed it, but the consensus was that it was far above the heads of those it hoped to please. In Germany, there was the book by Hilbert and Cohn–Vossen (1952).

But the 20th century witnessed a gradual collapse of geometry. Favored topics became arithmetic and number theory; they have ancient roots, are one of the top fields in today's mathematical research, and include large portions that are independent of the messy rest of mathematics. Therefore, they are central to many charismatic teachers' efforts to fire youngsters' imaginations towards mathematics.

2.3 The 1960s and the new math approach to mathematics education

Far bolder than those half-hearted attempts to enrich the highly endowed students with properly modern topics was the second broad approach to mathematics education exemplified by the new math of the 1960s.

Militantly anti-historical, I viewed the state of mathematics in the 1960s, and the direction in which it was evolving at that particular juncture in history, as an intrinsic product of historical necessity. This is what made it a model at every level of mathematics education. If the research frontier of the 1960s had *not* been historically necessary, new math would have lost much of its gloss or even legitimacy. The evidence, however, is that the notion of historical necessity as applied to mathematics (as well as other areas!) is merely an ideological invention. This issue is important and tackled at length in Chapter 4 of this volume.

In any event, new math died a while ago, victim of its obvious failure as an educational theory. The Romans used to say that "of the dead, one should speak nothing but good." But the new math's unmitigated disaster ought at least teach us how to avoid a repetition. However, it is well known that failure is an orphan (while success has many would-be parents), that is, no responsibility for this historical episode is claimed by anyone, as of today.

Take for example the French formalists who once flourished under the pen-name of Bourbaki (I shall have much more to say about them). They nurtured an environment in which new math became all but inevitable, yet today they join everyone else in making fun of the outcome, especially when it hurts their own children or grandchildren. This denial of responsibility is strikingly explicit in a one-hour story a French radio network devoted to the Bourbaki a few years ago. (Audio-cassettes may be available from the Société Mathématique de France.) One hears in it that the Bourbaki bear no more responsibility than the French man in the street (failure is indeed an orphan), and that they have never made a statement in favor of new math. On the other hand, having paid attention while suffering through the episode as the father of two sons, I do not recall their making a statement against new math, and I certainly recall the mood of that time.

Be that as it may, it is not useful to wax indignant, but important to draw a lesson for the future. The lesson is that *no frontier mathematics research must again be allowed to dominate mathematics education*. At the other unacceptable extreme, needless to say, I see even less merit in the notion that one can become expert at teaching mathematics or at writing textbooks, yet know nothing at all about the subject. Quite to the contrary, the teachers and the writers must know a great deal about at least some aspects of mathematics.

Fortunately, mathematics is not the conservatives' ivory tower. As will be seen in Chapter 4, I see it as a very big house that offers teachers a rich choice of topics to study and transmit to students. The serious problem is how to choose among those topics. My point is that this choice must not be left to people who have never entered the big house of mathematics, nor to the leaders of frontier mathematics research, nor to those who claim authority to interpret the leaders' preferences. Of course, you all know already which wing of the big house I think deserves special consideration. But let me not rush to talk of fractals, and stop to ask why the big house deserves to be visited.

3 The purely utilitarian argument for widespread literacy in mathematics and science

My own experiences suggest, and all anecdotal reports confirm, that traditional mathematics (of the kind described in the section before last) does marvels when a very charismatic teacher meets ambitious and mathematically gifted children. Helping the very gifted and ambitious is an extraordinarily important task, both for the sake of those individuals and of the future development of math and science. But (as already stated) I also believe that math and science literacy must extend beyond the very gifted pupils.

Unfortunately, as we all know, this belief is not shared by everyone. How can we help it become more widely accepted? All too often, I see the need for math and science literacy referred to exclusively in terms of the needs (already mentioned) of future math and science teaching and research, and those of an increasingly technological society. To my mind, however, this direct utilitarian argument fails on two accounts: it is not politically effective; and it is not sufficiently ambitious.

First of all, if scientific literacy is valuable and remains scarce, it has always been hard to explain why the scientifically literate fail (overall) to reap the financial rewards of valuable scarcity. In fact, scientific migrant workers, like agricultural ones, keep pouring in from poorer countries. Recent years were especially unkind to the utilitarian argument since many engineers and scientists are becoming unemployed and had to move on to fields that do not require their specialized training.

Even though this is an international issue, allow me to center the following comments on the conditions in the USA. In its crudest form, very widespread only a few years ago, the utilitarian argument led many people to compare the United States unfavorably to countries, including Russia, France, or Japan, with far more students in math or science. Similarly unfavorable comparisons concerned foreign language instruction in the USA to that in other countries. The explanation in the case of the languages of Hungary or Holland is obvious: the Hungarians are not genetically or socially superior to the Austrians, but the Austrians speak German, a useful language, while Hungarian is of no use elsewhere; hence, multilingual Hungarians receive unquestioned real-life rewards. Similarly, school programs heavy in compulsory math are tolerated in France and Japan because they provide unquestioned great real-life rewards to those who do well in math.

For example, many jobs in France that require little academic knowledge to be performed are reserved (by law) for those who pass a qualifying examination. The exam seeks objectivity, and ends up being heavy on math. There are many applicants, the exams are difficult, and the students are motivated to be serious about preparing for them.

Some of these jobs are among the best possible. For example, in many French businesses one cannot approach the top unless one started at the Ecole Polytechnique, the school I attended. (I first entered Ecole Normale Supérieure, but left immediately). For a time after Polytechnique was founded (in 1794), it first selected and judged its students on the broad and subjective grounds ideally used in today's America, but later the criteria for entrance and ranking became increasingly objective—that is, mathematical. One reason was the justified fear of nepotism and political pressure, another the skill of Augustin Cauchy (1789–1857), a very great mathematician and also a master at exerting self-serving political pressure.

The result was clear at the forty-fifth and fiftieth reunions of my class at Polytechnique. For a few freshly retired classmates a knowledge of science had been essential. But most had held very powerful positions to general acclaim, yet hardly remembered what a complex number is—because it has not much mattered to them. They gave no evidence of an exceptionally strong love of science. (I do not know what to make of the number of articles our *Alumni Monthly* devotes to the paranormal.) But my classmates could never have reached those powerful positions without joining the Polytechnique "club"; to be a wizard at math, at least up to age twenty, was part of the initiation and a desired source of homogenity.

The United States of America also singles out an activity that brings monetary rewards and prestige that continue through a person's life—independent of the person's profession. This activity is sports. In France it is math. For example, one of my classmates (Valéry Giscard d'Estaing) became President of France, his goal since childhood; to help himself along, he chose to go to a college even more demanding than MIT.

For a long time France recognized a second path to the top: a mastery of Greek or Latin writers and philosophers. But by now this path has been replaced by an obstacle course in public administration. A competition continues between the two ways of training for the top, but no one claims that either mathematics or the obstacle courses is important *per se*. You see how little bearing this French model has on the situation in the USA.

Needless to say, many French people have always complained that their school system demands more math than is sensible; other French people complain that the teaching of math is poor. And I heard the same complaints on a trip to Japan. So my feeling is that the real problem may not involve embarrassing national comparisons.

4 In praise of widespread literacy in mathematics and science

Lacking the purely utilitarian argument, what could one conceivably propose to justify more and better math and physics? When I was young some of my friends were delighted to reserve real math to a small elite. But other friends and I envied the historians, the painters, and the musicians. Their fields also involved elite training, yet their goals seemed blessed by the additional virtue of striking raw nerves in other human beings. They were well understood and appreciated by a wide number of people with comparatively minimal and unprofessional artistic education. To the contrary, the goals of my community of mathematicians were becoming increasingly opaque beyond a circle of specialists. Tongue in cheek, my youthful friends and I dreamt of some extraordinary change of heart that would induce ordinary people to come closer to us of their own free will. They should not have to be bribed by promises of jobs and money, as was the case for the French adolescents. Who can tell, a popular wish to come closer to us might even induce them to buy tickets to our performances!

When our demanding dream was challenged as ridiculous and contrary to history and common sense, we could only produce one historical period when something like our hopes had been realized. Our example is best described in the following words of Sir Isaiah Berlin (Berlin 1979):

"Galileo's method ... and his naturalism, played a crucial role in the development of seventeenth-century thought, and extended far beyond technical philosophy. The impact of Newton's ideas was immense: whether they were correctly understood or not, the entire program of the Enlightenment, especially in France, was consciously founded on Newton's principles and methods, and derived its confidence and its vast influence from his spectacular achievements. And this, in due course, transformed—indeed, largely created—some of the central concepts and directions of modern culture in the West, moral, political, technological, historical, social—no sphere of thought or life escaped the consequences of this cultural mutation. This is true to a lesser extent of Darwin .... Modern theoretical physics cannot, has not, even in its most general outlines, thus far been successfully rendered in popular language as Newton's central doctrines were, for example, by Voltaire."

Voltaire was, of course, the most celebrated French writer of the eighteenth century and mention of his name brings to mind a fact that is instructive but little known, especially outside France: it concerns the first translation of Newton into French, which appeared in Voltaire's time. Feminists, listen: the translator was Gabrielle Émilie le Tonnelier de Breteuil, marquise du Châtelet-Lomont (1706–49). Madame du Châtelet was a pillar of High Society: her salon was among the most brilliant in Paris.

In addition, the XVIIIth century left us the letters that the great Leonhard Euler (1707–83) wrote to "a German princess" on topics of mathematics. Thus a significantly broad scientific literacy was welcomed and conspicuously present in a century when it hardly seemed to matter.

5 Contrasts between two patterns for hard scientific knowledge: Astronomy and history

Why is there such an outrageous difference between activities that appeal to many (like serious history), and those which only appeal to specialists? To try and explain this contrast, let me sketch yet another bit of history, comparing knowledge patterned after astronomy and history.

The Ancient Greeks and the medieval scholastics saw a perfect contrast between two extremes: the purity and perfection of the Heaven, and the hopeless imperfection of the Earth. *Pure* meant subject to rational laws which involve simple rules yet allow excellent predictions of the motion of planets and stars. Many civilizations and individuals believe that their lives are written up in full detail in a book and hence can in theory be predicted and cannot be changed. But many others (including Ancient Greeks) thought otherwise. They expected almost everything on Earth to be a thorough mess. This allowed events that were in themselves insignificant to have unpredictable and overwhelming consequences—a rationalization for magic and spells. This *sensitive dependence* became a favorite theme of many writers; Benjamin Franklin's *Poor Richard's Almanac* (published in 1757), retells an ancient ditty as follows:

> "A little neglect may breed mischief.
> For lack of a nail, the shoe was lost;
> for lack of a shoe, the horse was lost;
> for lack of a horse, the rider was lost;
> for lack of a rider, the message was lost;
> for lack of a message, the battle was lost;
> for lack of a battle, the war was lost;
> for lack of a war, the kingdom was lost;
> and all because of one horseshoe nail."

From this perspective, it seems to me that belief in astrology, and the hopes that continue to be invested today in diverse would-be sciences, all express a natural desire to escape the terrestrial confusion of human events and emotions by putting them into correspondence with the pure predictability of the stars.

The beautiful separation between pure and impure (confused) lasted until Galileo. He destroyed it by creating a terrestrial mechanics that obeyed the same laws as celestial mechanics; he also discovered that the surface of the Sun is covered with spots and hence is imperfect. His extension of the domain of order opened the route to Newton and to science. His extension of the domain of disorder made our vision of the universe more realistic, but for a long time it removed the Sun's surface from the reach of quantitative science.

After Galileo, knowledge was free from the Greeks' distinction between Heaven and Earth, but it continued to distinguish between several levels of knowledge. At one end was *hard* knowledge, a science of order patterned after astronomy. At the other end, is *soft* knowledge patterned after history, i.e., the study of human and social behavior. (In German, the word *Wissenschaft* stands for both *knowledge* and *science*; this may be one of several bad reasons why the English and the French often use *science* as a substitute for *knowledge*.)

Let me at this point confess to you the envy I experienced as a young man, when watching the hold on minds that is the privilege of psychology and sociology, and of my youthful dreams of seeing some part of hard science somehow succeed in achieving a similar hold. Until a few decades ago, the nature of science made this an idle dream. Human beings (not all, to be sure, but enough of them) view history, psychology, and sociology as alive (unless they had been smothered by mathematical modeling). Astronomy is not viewed as alive; the Sun and the Moon are superhuman because of their regularity, therefore gods. In the same spirit, many students view math as cold and dry, something wholly separate from any spontaneous concern, not worth thinking about unless they are compelled. Scientists and engineers must know the rules that govern the motions of planets. But these rules have limited appeal to ordinary humans because they have nothing to do with history or the messy, everyday life, in which, let me repeat, the lack of a nail can lose a horse (a battle, a war, and even a kingdom) or a bride.

6 A new kind of science: Chaos and fractals

Now we are ready for my main point. In recent years the sharp contrast between astronomy and history has collapsed. We witness the coming together, not of a new *species* of science; nor even (to continue in taxonomic terms) a new *genus* or *family*, but a much more profound change. Towards the end of the 19th century, a seed was sowed by Poincaré and Hadamard; but practically no one paid attention, and the seed failed to develop until recently. It is only since the 1960s that the study of true disorder and complexity has come onto

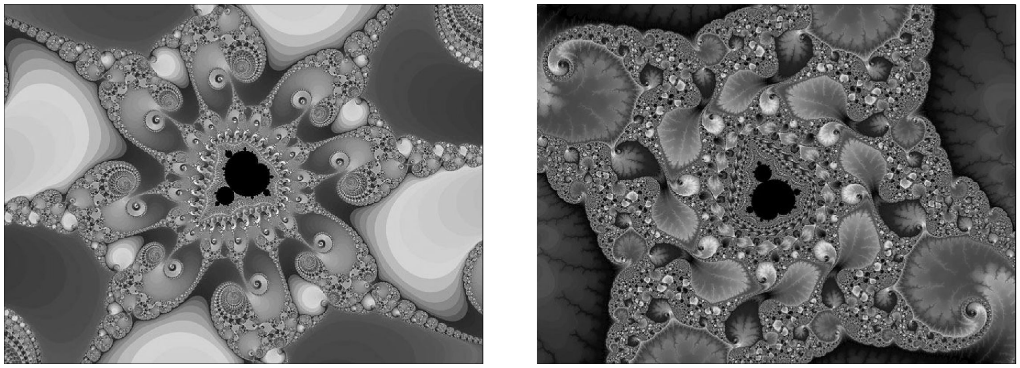

Figure 1: Two close-up views of parts of the Mandelbrot set.

the scene. Two key words are *chaos* and *fractals*, but I shall keep to *fractals*. Again and again my work has revealed cases where simplicity breeds a complication that seems incredibly lifelike.

The crux of the matter is a geometric object that I first saw in 1979, took very seriously, and worked hard to describe in 1980. It has been named the *Mandelbrot set*. It starts with a formula so simple that no one could possibly have expected so much from it. You program this silly little formula into your trusty personal computer or workstation, and suddenly everything breaks loose. Astronomy described simple rules and simple effects, while history described complicated rules and complicated effects. Fractal geometry has revealed simple rules and complicated effects. The complication one sees is not only most extraordinary but is also spontaneously attractive, and often breathtakingly beautiful. See Figure 1. Besides, you may change the formula by what seems a tiny amount, and the complication is replaced by something altogether different, but equally beautiful.

The effect is absolutely like an uncanny form of white magic. I shall never forget the first time I experienced it. I ran the program over and over again and just could not let it go. I was a visiting professor at Harvard at the time and interest in my pictures immediately proved contagious. As the bug spread, I began to be stopped in the halls by people who wanted to hear the latest news. In due time, the *Scientific American* of April 1985 published a story that spread the news beyond Harvard.

The bug spread to tens, hundreds, and thousands of people. I started getting calls from people who said they loved those pictures so much that they simply had to understand them; where could they find out about the multiplication of complex numbers? Other people wrote to tell me that they found my pictures frightening. Soon the bug spread from adults to children, and then (how often does this happen?) from the kids to teachers and to parents.

Lovable! Frightening! One expects these words to be applied to live, warm bodies, not to mere geometric shapes. Would you have expected kids to go to you, their teachers, and ask you to explain a mathematical picture? And be eager enough to volunteer to learn more and better algebra? Would you expect strangers to stop me in a store downtown, because they just have to find out what a complex number is?

Next, let me remind you that the new math fiasco started when a committee of my elders, including some of my friends, all very distinguished and full of goodwill, figured out among themselves that it was best to start by teaching small kids the notions that famous professors living in the 1950s viewed as being fundamental, and therefore simple. They wanted grade schoolers to be taught the abstract idea of a set. For example, a box containing five nails was given a new name: it became a set of five nails. As it happened, hardly anyone was dying to know about five-nail sets.

On the other hand, the initial spread of fractals among students and ordinary people was neither planned nor supported by any committee or corporation, least of all by IBM, which supported my scientific work but had no interest in its graphic or popular aspects. This spread was one of the most truly spontaneous events I ever heard of or witnessed. People could not wait to understand and master the white magic and find out about those crazy Mandelbrot sets. The five-nail set was rejected as cold and dry. The Mandelbrot set was welcomed almost as if it were alive. Everything suggests that its study can become a part of liberal knowledge!

Chaotic dynamics meets the same response. There is no fun in watching a classical pendulum beat away relentlessly, but the motion of a pendulum made of two hinged sticks is endlessly fascinating. I believe that this contrast reveals a basic truth that every scientist knows or suspects, but few would concede. The only trace of historical necessity in the evolution of science may be that its grand strategy is to begin with questions that are not necessarily the most exciting, but are simple enough to be tackled at a given time.

The lesson for the educator is obvious. Motivate the students by that which is fascinating, and hope that the resulting enthusiasm will create sufficient momentum to move them through material that must be studied but is less widely viewed as fun.

7 Just beyond the easy fractals lurk overwhelming challenges

This last word, "fun," deserves amplification. The widely perceived difficulty of mathematics is a reason for criticism by the outsider. But for the insider it is a source of pride, and mathematics is not viewed as real unless it is difficult. In that sense, fractal geometry is as real as can be, but with a few uncommon wrinkles.

The first uncommon wrinkle has already been mentioned: hardly any other chapter of mathematics can boast that even to the outsider its first steps are fun.

Pushing beyond the first steps, a few additional ones led me (and soon led others beyond counting) to stunning observations that the eye tells us must be true, but the mind tells us must be proven.

A second uncommon wrinkle of fractal geometry is that those observations are often both simple and new; at least, they are very new within recent memory. Hardly any other chapter of mathematics can boast of simple and new observations worth making. Therefore, fractal geometry has provided multitudes with the awareness that the field of mathematics is alive.

A third uncommon wrinkle of fractal geometry is that, next to simple and new observations that were easy to prove, several revealed themselves beyond the power of the exceptionally skilled mathematicians who tackled them. Thus, some of my earliest observations about the Mandelbrot set remain open. Furthermore, no one knows the dimensions of self-affine sets beyond the simplest. In physics, turbulence and fractal aggregates remain mysterious. The thrills of frontier life can be enjoyed right next to the thriving settlements. Hardly any other chapter of mathematics can boast of so many simple but intractable conjectures.

A fourth wrinkle concerns the easy beginnings of fractal geometry. Thanks to intense exposure, it is quite true that much about fractals appears obvious today. But yesterday the opposite view was held by everyone. My writings have—perhaps with excessive verve—blamed mathematicians for having boxed themselves and everyone else in an intellectual environment where constructions now viewed as *proto-fractal* were once viewed as *pathological* and anything but obvious. This intellectual environment was proud of having broken the connections between mathematics and physics. Today there is a growing consensus that the continuity of the links between mathematics and physics is obvious, but the statements ring false in the mouths of those who denied and destroyed this continuity; they sound better in the mouths of those who rebuilt it.

To conclude this section, fractals may be unrepresentative. This is not a drawback but rather a very great strength from the viewpoint of education. If it is true that "math was never like that," it is also true that "this is more lifelike than any other branch of math."

8 The computer is the teacher's best friend in communicating the meaning of rigor

One passionate objection to the computer as the point of entry into real mathematics is the following: if the young replace solving traditional problems by computer games, they will never be able to understand the fundamental notion of mathematical rigor. This fear is based on an obvious chain of associations: the computer started as a tool of applied mathematics, applied mathematicians spurn rigor, the friend of my enemies is my enemy, therefore, the computer is the enemy of rigor.

With equal passion I think that the precise contrary is true: *rain or shine, the computer is rigor's only true friend*. True, a child can play forever with a ready-made program that draws Mandelbrot sets and never understand rigor, nor learn much of any value. But neither does the child who always does his mathematical homework with access to the teacher's answer book. On the contrary, the notion of rigor is of the essence for anyone who has been motivated to write a computer program—even a short one—from scratch.

When I was a student a non-rigorous proof did not scream *look out* at me and I soon realized that even my excellent teachers occasionally failed to notice clearcut errors in my papers. In the case of a computer program, on the contrary, being rigorous is not simply an esthetic requirement; in most cases, a non-rigorous program fails completely, and the slightest departure from absolute rigor makes it scream "Error!" at the programmer. No wonder that the birth of the computer was assisted by logicians and not mainstream mathematicians. (This topic is discussed in Mandelbrot 1993a.) It is true that, on occasion, a nonrigorous program generates meaningless typography or graphics, or—worse—sensible-looking output that happens to be wrong. But those rare examples only prove that programming requires no less care than does traditional proof.

Moreover, the computer programmer soon learns that a program that works on one computer, with its operating system, will not work on another. He will swear at the discrepancies, but I cannot imagine a better illustration of the changeability and arbitrariness of axiomatic systems.

Many other concepts used to be subtle and controversial before the computer made them become clear. Thus, computer graphics refreshes a distinction between fact and proof, one that many mathematicians prefer *not* to acknowledge but that Archimedes described wonderfully in these words: "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method has previously given us some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems that the volumes of a cone and a pyramid are one-third of the volumes of the cylinder and prism (respectively) having the same base and equal height, the proofs of which Eudoxus was the first to discover, no small share of the credit should be given to Democritus who was the first to state the fact, though without proof."

The first two sentences might easily have been written in our time by someone describing renascent experimental mathematics, but Archimedes lived from 287 to 212 BC, Democritus from 460 to 370 BC and Eudoxus from 408 to 355 BC. (Don't let your eyes glaze over at the names of these Ancient heroes. This chapter is almost over.)

When a child (and why not an adult?) becomes tired of seeing chaos and fractal games as white magic and draws up a list of observations he wants to really understand, he goes beyond playing the role of Democritus and on to playing the role of Eudoxus. Moreover, anyone's list of observations is bound to include several that are obviously mutually contradictory, stressing the need for a referee. Is there a better way of communicating another role for rigor and a role for further experimentation?

9 Conclusion

As was obvious all along, I am a working scientist fascinated by history and education, but totally ignorant of the literature of educational philosophy. I hope that some of my thoughts will be useful, but many must be commonplace or otherwise deserve to be credited to someone. One area where I claim no perverse originality is the historical assertions: they are documented facts, not anecdotes made up to justify a conclusion.

Now to conclude. The best is to quote myself and to ask once again: Is it extravagant to hope that, starting with this piece of mathematics called fractal geometry, we could help broaden the small band of those who see mathematics as essential? That band ought to include every educated citizen and therefore to have mathematics take its place among the liberal arts. A statement of hope is the best place to close.

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Chapter 4

Mathematics and Society in the 20th Century[1]

Benoit B. Mandelbrot

Mathematics education and research are two separate crafts, but—for practical as well as intellectual reasons—it is best if they know each other. In particular, it is very important for mathematics educators to have a broad and balanced view of the way research mathematicians perceive their craft. They must realize that the perception has kept changing throughout history and never as sharply as in the 20th century. This chapter's goal is to recount a few highly significant features of the strife that came in the preceding hundred years. Mathematics ended that century in great spirit and in a state of great vigor, renewed collegiality and marvelous diversity.

But in the 1960s and 1970s, the representatives of the profession described the flow of 20th century mathematics as that of a single majestic river whose irresistible course was not touched by historical accident but had been preordained by inner logic. It necessarily proceeded inevitably and inexorably towards increasingly general, structural, or fundamental notions—which happened to be increasingly abstract. In the spirit of "the end of history," the descriptions never referred to the past or the messiness of Earth.

The majestic flow in question was unflinchingly understood to be leaving aside many people (including myself), and innumerable topics that concern either the foundations (logic) or the applications. We were told that much of what *looks* like mathematics is *not really* mathematics, even though the distinction may not be obvious to the outsider.

The position I am about to describe is starkly different. I believe and I hope to convince you that mathematics is *not* the conservatives' ivory tower. It is a very big house on a rolling terrain, with many doors, windows open to many horizons and bridges to many other houses.

It need not be the Queen of Court Etiquette in Science looking down on most of her subjects from an ivory tower up on a high hill. It deserves to be the beloved Queen of all the Scientists' Hearts, and of the Soul of Science, the only non-contrived link that could prevent various parts from scattering away from one another.

Compared to the conservative view of mathematics, mine is far broader and far more strongly linked to other human activities. It is also a more diverse and lively subject. In particular, it is attractive to persons who are not professional research mathematicians, a category that includes students and most teachers of mathematics. My strong opinions represent a minority view, but one that is increasingly widely shared and I have no doubt will prevail.

In any event, my interpretations and opinions are neither capricious nor based on idle rumor or anecdote, but on widely ranging reading, active and uninterrupted participation in events that occurred in the USA and France over fifty years, and reports by an uncle who was a prominent mathematician in Paris and Houston and participated in the immediately preceding thirty-five year period.

I see mathematical science as a very broad enterprise that shelters many diverse topics, ranging from the very concrete to the very abstract. This view is well represented by a simile I heard used by Hermann Weyl (1885–1955). He compared mathematics to the delta of a great river, one made of many streams: they may vary in their width and the speed of the flow through them; nevertheless, all are always a part of the system, and no individual stream is permanently the most important. This simile represented the mood of mathematics close to the year 1900—and also, for that matter, its mood near the year 1800. More importantly, mathematics has been changing so fast for a decade or so that I feel that Weyl's simile became applicable again in the year 2000.

But the resemblances between these snapshots taken centuries apart certainly do not imply that mathematics is unchanging, something outside ordinary history. In mathemat-

[1] Adapted from an invited address "What will remain of 20th century formal science" at the Europeäisches Forum 1992, held in Alpbach, Austria. This text remains self-contained and preserves some of its original flavor, in part by repeating some points that were already made elsewhere in this book but bear emphasis.

ics, as in every other aspect of human life, the 20th century gave us an example of something starkly different: a rocky history and continuing conflict. Mathematics was not ruled by its own determinism; it did not evolve separately from every other aspect of human knowing and feeling; it has on the contrary been profoundly affected by endless external vicissitudes.

The words *profoundly affected by* must not be misunderstood as meaning *enslaved by*. Of all the triumphs of humanity, the discovery and the development of mathematics is perhaps the greatest kind. A field's importance to the overall human experience is necessarily reflected by the role that internal logic has upon its development; nevertheless, strife has been present in mathematics since the Ancient Greeks. We shall see this when this story ends by mentioning the long-standing conflict between the traditions of Plato, the ideologue, and Archimedes, the experienced scientist. Like every individual human activity, mathematics very much participates in general history, politics, demography, and technology, and it is heavily influenced by the idiosyncrasies of a few key people. Let me give some examples from this century.

Around 1920, a group of Polish mathematicians collected around a very forceful man named Waclaw Sierpinski (1882–1969). They chose to concentrate on a field that was not practiced much in the reigning intellectual capitals, and founded a very abstract new branch often called *Polish mathematics*. They proudly proclaimed that their goal involved national politics: they did not want the newly reestablished Poland to become a mathematical satellite of Paris or Göttingen. I know that Providence is credited with working in mysterious ways. Yet, would anyone claim that Polish nationalism after more than a century of partition had anything to do with the historical determinism of mathematics? Polish mathematics became an important force pushing towards abstraction at all cost. Yet, by a bitter irony, some of the notions it originated failed to become important in mathematics, but eventually became important to physics—through fractal geometry.

My second example concerns Godfrey Harold Hardy (1877–1947), a strong person as well as a strong and highly inventive mind. The Poles had no strong native physics to contend with, but British mathematics of Hardy's youth was dominated by a form of mathematical physics that was extraordinarily effective (the Heaviside Calculus differentiated discontinuous functions!) but had little concern with continental rigor. During World War I, Hardy was an outspoken pacifist who recoiled from the practical uses of this old British mathematics. During another War, he wrote (Hardy (1940)), an impassioned account of his ideal of pure mathematics. For him, good mathematics could have no bad application—for the simple reason that it could have *no* application of any sort. By another bitter irony, his best example of total inapplicability turned out, in due time, to be essential to a problem he would have loathed: cryptography.

A three-page review of Hardy (1940) in the famous weekly *Nature* by the Nobel-winning chemist Frederick Soddy begins "This is a slight book. From such cloistral clowning the world sickens . . . 'Imaginary' universes are so much more beautiful than this stupidly constructed 'real' one, according to the author . . . Most scientists, however, still believe that . . . the real universe . . . is not stupidly constructed." But nothing can break the appeal of a tract that discriminates between the good and the bad without hesitation. Hardy's book remains in print and continues to this day to enchant some of the young. But would anyone claim that Hardy's militant anti-nationalism had anything to do with the historical determinism of mathematics?

From ideology, let us move on to demography. The 1910s were very cruel to French mathematics. First, Henri Poincaré (1854–1912) died prematurely on the operating table, then millions of young people died in trench warfare, and finally—perhaps worst of all—millions returned broken in health or spirit to a country that did not dare make heavy demands on them. As a result, the young postwar French mathematicians of the 1920s found that the only available teachers were men who had already been ill or old in 1914 and so did not go to war. Some have written movingly about the hardship of training without the usual parental supervision from slightly older advisors, and (as may have been expected) this hardship contributed to the emergence of several very strong personalities. In any event, the France of the late 1920s and the 1930s gave rise to an extremist movement calling itself Bourbaki. But would anyone claim that a demographic unbalance in a country with a long and glorious mathematical tradition has anything to do with the historical determinism of mathematics?

André Weil (1906–1994), now acknowledged as the mind behind Bourbaki, observed late in life that in his prime years, mathematics was little influenced by physics. Was that a natural feature of the preordained development of mathematics? Or could it be that Weil's views were set even before a visit to Göttingen in the 1920s? David Hilbert's dream Mathematics Institute there had three parts: a very pure one that Weil worshipped, one on numerical methods and one on mathematical physics. In the latter part, Max Born and Werner Heisenberg were in the process of creating quantum mechanics—but Weil apparently did not notice.

From demography, let us move to another form of ideology. Soviet anti-semitism treated Jewish mathematicians harshly; Jewish physicists, less so. Hence a number of very gifted mathematicians transferred to physics institutes, where they were welcome. Their move contributed greatly to the formation of the current very rigorous form of mathematical physics. Would anyone claim that Soviet ethnic politics have anything to do with the historical determinism of mathematics?

No one would claim that the specific historical determinism of mathematics only reflected the intellectual moods and fashions that rule society at large. But it happens that a very unusual mood prevailed early in this century, particularly in the 1920s. One especially visible and durable effect was the invention of the International Style in architecture, with its heavy emphasis on structure. In Finland, the very unusual small country where this style was born, modern architecture merged smoothly into what came before it, without discontinuity and without heavy dogmatism. But modern architecture became dogmatic in Germany with the Bauhaus and in France with Le Corbusier (1887–1965). The latter built few houses but made many sketches (for example, his proposed ideal improvement of Paris evokes the worst present suburbs of Moscow). When I was young, Le Corbusier was billed as a great intellect to whom modern architecture owed its intellectual legitimacy. Indeed, he wrote a great deal, but I find little in his writings beyond sophomoric trash. It may be that Bauhaus was useful, even commercially inevitable at a certain stage of the technology and economics of raising large buildings, but no one ever convinced me that they were an inevitable intellectual wave of the future.

Think also of physics. Having confirmed existence of the atom in the 1900s, it went on to focus increasingly on the search for the most fundamental structural components of matter, increasingly tiny ones. Biology took this path later.

How was mathematics affected by the above-mentioned politics, demographics, and general intellectual moods? I view them all as responsible for the fact that near the middle of our century mathematics behaved in ways totally at variance with its mood today and its mood in 1900 or 1800.

This atypical mathematics is conveniently denoted by the name it took in France, but the current that gave rise to Bourbaki also affected many countries other than Britain, France and Poland. It strongly affected the USA, with a little-known wrinkle. One might have expected a brash new industrial giant to favor applications, but in terms of mathematical research the precise contrary was true. In Europe, the 19th century had created wide-ranging establishments against which Bourbaki could revolt. In the USA, before the arrival of refugees from Stalin and Hitler, research mathematics was dominated by aristocrats and anarchists, hence was very pure (as well as outstanding on its terms). Bourbaki did not reach the outlying countries Sweden and Finland, and there were strong counteracting forces in Germany and Russia. In the 1960s, when Bourbaki was its strongest, it benefited from another extraneous event: Sputnik created a period of unprecedented economic growth in Academia, with minimal social pressure on the sciences, and greatly increased the number of math PhDs, including many Bourbaki products. The math departments' balance was overwhelmed by them.

To sum up, Bourbaki found roots by selecting one of the many components of the mathematics of 1875–1925, gathered strength during the second quarter of our century (the period to which the above examples refer), and took power around 1950. During the third quarter of the century it exerted an extraordinary degree of control. There was no disorder in mathematics, but the field was narrowed down to a truly extraordinary extent. At one time it seemed to reduce to little more than algebraic topology; at a later time, to number theory and algebraic geometry. These are extremely important fields, to be sure, but concentration on a single field was quite contrary to the historical tradition that I have already mentioned and that had led Hermann Weyl to the image of the delta of the Nile. Mathematics seemed to have reduced itself to basically a single stream at any given time. This happened to be the cliché description that Herman Weyl (in a contrasting image) applied to physics.

The Bourbaki, as has already been implied, never paid attention to the historical accidents that contributed to their birth; they felt themselves to be the necessary and inevitable response to the call of history. Today, however, this call seems forgotten, and there is wide consensus that, like new math, "Bourbaki is dead."

Who killed Bourbaki? Throughout its heyday, my friend Mark Kac (1914–84) and many other open-minded mathematicians argued, in vehement speeches and articles, that Bourbaki had misread mere accidents for the arrow of history. But such negative criticism invariably lacked bite, and it had no effect. My own partisan opinion is that Bourbaki's fate was typical of many ideologies outside science. The founders could only insure their immediate succession; gradually, the ideological fervor weakened and the movement continued largely by force of habit. The resulting weakening was gradual and not obvious. But everyone noticed when the movement was knocked down by yet another event that had nothing to do with the historical determinism in mathematics. This event was something I view as a return to sanity, namely the rebirth of experimental mathematics that followed (slowly, as we shall see) the advent of the modern computer.

From where did the computer come? From the mathematical sciences understood in a broad sense. What relation is there between the advent of the computer and mathematics as narrowly reinterpreted by Bourbaki? None whatsoever. The computer arose from the convergence of two fields that surely belong to mathematics but were spurned by Bourbaki, namely, logic and differential equations. We all know that one must never rewrite history as it might have proceeded if two crucial events had chanced to occur in the reverse order. But in this instance the temptation is strong to air the following conviction. Had an earlier arrival of the computer saved experimental mathematics from falling into a century of decline, Bourbaki might have never seemed to anyone to be an unavoidable development. Let me elaborate on the computer's roots.

Surprisingly, while **Foundations of Analysis** was (for a while) the overall title of their treatises, the Bourbaki had only contempt for the logical foundations of mathematics, as in the work of Kurt Gödel (1906–78) and Alan Turing (1912–54). In the 1930s, Turing had phrased his model of a logical system in terms of an idealized computer. His *Turing machine* had a very great influence on the thinking of those who developed the actual hardware.

However, the man who made the computer into a reality was John von Neumann (1903–57). He was not only a mathematician, but also a physicist and an economist, and his great breadth of interests came to include a passion to find ways to predict the weather.

Thus the computer was born in the 1940s from a strange combination of abstract logic and the desire to control Nature. Eventually, the computer changed mathematics in a very profound fashion. But for a very long time, core mathematicians felt totally unconcerned, and viewed it with revulsion. Because of his work on the computer, von Neumann ceased to be accepted as a mathematician and in 1955 he decided to resign from the Princeton Institute for Advanced Study. (He died before his planned move to California.)

The year 1955 was also the date of publication of Fermi, Pasta & Ulam (1955), a text that appeared only in a Los Alamos report but was widely read and viewed as an early masterpiece of experimental mathematics before it was actually printed in Fermi (1965) (pp 977–988) and then in Ulam (1974) (pp 490–501). A comment by Stanislaw Ulam (1909–1984) informs us that the initiative for using the computer to assist mathematical research had come from Enrico Fermi (1901–1954), who was of course a physicist, not a mathematician. And Ulam asserts that "Mathematics is not really an observational science and not even an experimental one. Nevertheless, computations were useful in establishing some rather curious facts about simple mathematical objects." Surprised, I reached for a more positive statement in the autobiography, Ulam (1976), but found nothing worth quoting.

How did experimental mathematics fare during the 25 years after 1955? That period happens to end in the year of publication of Mandelbrot (1980), and coincided with the heyday of Bourbaki. In a near-perfect first approximation, it saw *no experimental mathematics at all*. Not only was the lead of von Neumann and Fermi not followed by mathematicians, but their disinterest for the computer was carefully considered, not caused by ignorance. For example, when I was new at IBM, which I joined in 1958, opportunities to use computers were knowingly and systematically offered to every mathematician with a good name who could be coaxed into the building. Not one of them paid attention to the offer. Interest in experiment did not spread to at least some mathematicians until my work started attracting wide attention.

In understanding the process of discovery, the slowness of the acceptance of the computer brings up forcibly a very old issue: the respective contributions of the tool and of its user. Galileo wrote a whole book complaining bitterly about those who belittled him by claiming that his discovery of sunspots was only due to his having lived during the telescope revolution. In fact, telescopes were widespread but useless before one reached Galileo's steady hand and good eye. For contrast, consider the chapter of mathematics called the global theory of iteration of rational functions, to which the Mandelbrot set belongs. Pierre Fatou (1878–1929) and Gaston Julia (1893–1978) are—quite rightly—praised for developing this chapter, and no one would dream of belittling their contributions as being due to their having lived during the age of Paul Montel (1876–1975). Montel was the mathematician who, in 1912, discovered Fatou's and Julia's key tool, called the normal families of functions. Soon afterwards he was called into the Army, leaving behind Fatou (who was a cripple) and Julia (who had come back from the trenches as a wounded war hero). After World War I, Montel looked after the theory of iteration as *his* baby. Today, in the noise that accompanies changes in mathematics, those who use the computer are treated like a Galileo and not like a Montel. That is, critics are found to belittle their work as solely due to their living in the computer age. If it were so, experimental mathematics would have thrived after von Neumann and Fermi; the preceding remarks show that it did not.

Let me summarize, make a general comment, and conclude. One cannot disregard the lessons of history, contrary to the belief of those who argued that the pure mathematics of the mid-20th century was preordained by destiny. Its birth in the 1920s was influenced by Polish and English ideology, a demographic catastrophe in France, and the general intellectual mood of the day; its success was hastened by a long spurt of economic growth, and its demise was hastened by a mere technological development. None of these events was influenced by mathematics, none was preordained, and none acted immediately. In any event, von Neumann's and Fermi's lead was not followed by other mathematicians.

To conclude, "What will remain of 20th century mathematics?" There can be no short and truthful answer, because at this point of its history, mathematics is in healthy and constructive turmoil. Once again, the Bourbaki utopia flourished when every science was experiencing unprecedented growth and minimal social pressure. It seemed that any would-be peer group could organize itself and prosper with no hindrance from other, equally self-interested peer groups. But today the sciences face scarcity and strong pressure to justify both their size and their goals, and everyone bemoans the absence of generalists capable of representing more than a few groups. How the effects of the resulting intractabilities and pressures will combine with the internal logic of mathematics, of the computer and also of today's mathematical physics— that thriving no man's land between theorem proving and observation of nature—is simply beyond prediction. Fortunately for the teachers of mathematics, they are not asked to predict, but it is best for them to know the past, if only to avoid being drawn to repeating its deep errors.