Mitchell Jay Feigenbaum

One morning, not long ago, it was about eight o'clock, I left my New York apartment on York Avenue for my daily walk, which that day led me along 63rd Street to Central Park, where I walked around for a while, then returned. I had reached the corner of First Avenue when I saw a familiar face, his hair a ragged mane. 'What are you doing here so late?' I inquired. He explained that he had run out of cigarettes, and also wanted to pick up the day's edition of *The New York Times* which he likes to read before going to bed. I wished him good night and walked on.

That was my good friend Mitch Feigenbaum. We live in the same apartment complex and have offices next to each other at the Rockefeller University. With luck we meet there in the late afternoon. The sketch of his life and work to date which is to follow is largely the result of numerous late evening talks with him.

Family

Mitch was born in Philadelphia on December 19, 1944, the son of Abraham Joseph and Mildred, née Sugar, both native New Yorkers. His father is the third of four children. His family hails from Loshitz, near Warsaw, from where Mitch's grandfather emigrated to the United States. Mitch thinks the family name was then Fajgenboim. His mother's father came to the US from Kiev. The name Sugar was made up by Ellis Island officials. She is the second of three children, she also has two half sisters.

Mitch's older brother Edward was decidedly a child prodigy who could read at a very early age. He is now a systems engineer and lives with his wife and two children outside Washington, DC. (He is not to be confused with a famous computer scientist who has the same first and last name.) Glenda, Mitch's younger sister, is an actuary at the Metropolitan Life Insurance Company and lives in New Jersey with her husband and two children.

After receiving a Master's degree in biology from New York University, Abraham Joseph was offered a university job in Rochester, New York, with a salary of $2000 a year. His wife thought that was not enough. Instead he became an analytical chemist at the Naval Shipyard in Philadelphia (we are now into the Second World War), where he had to study insecticides that could kill cockroaches on ships and took catering jobs on weekends to make some extra money.

In 1947—Mitch was two and a half—the family moved back to New York City, where they bought a two-story house in Brooklyn, renting out one floor. The father now obtained a position as analytical chemist with the New York Port Authority. Among his assignments was to test land where Idlewild (now J. F. Kennedy) Airport was to be built.

Mitchell calls his mother 'a physically and mentally powerful woman.' At age 14 she went to study at Hunter College but had to leave two years later in order to bring in money for her family. In her late teens she earned money by playing handball, at which she could beat men; and also by teaching remedial reading. Next she applied for and obtained a job in the knitting department of Lamston's department store in Spanish Harlem. At that point she knew neither Spanish nor how to knit, but quickly picked up both. Mitch characterizes her especially as an educator who served as his first role model. Later that switched to the father. 'I found his ways of thinking more substantial . . . I found him infinitely honest. He underexpressed himself, however, finding speaking in some ways superfluous.'

Early memories

Mitch was 'clean' when half a year old. He started talking 'reasonably late. Before that I just pointed at things, having my brother say what name they had.' His mother tried to teach him reading but he did not like that, in fact he did not read until he went to school. From early on he liked to know 'how things worked,' in the mechanical sense, liked to take things apart and to know 'what things looked like.' From age three he began to listen to music. 'This was very important to me.' He would be up by seven in the morning to listen to music. They had a radio. How does that one work? 'It struck me that a radio is remarkable because it has no phonograph records, and because 'radio waves' go through walls.'

From very early on he liked to sit and muse about things and observe, sitting on the porch. He remembers, at age four or five, calling out to his mother to come to the porch. 'Look at that woman. Why doesn't she fall?' It was a large-breasted lady; Mitch was wondering about her mechanical equilibrium.

From early on Mitch very much liked to draw, especially portraits. He discovered that drawing smooth, perfectly crafted curves could replace the rendering of minutiae. His drawings became more and more abstract, Miro-like. He stopped drawing at about age 21. 'In the end I had nothing new to express; it became cartoons. I did not know what to draw any more.'

School years

Mitch started school at age five, attending the public school PS208 for gifted children. The school had advanced programs. Spanish was taught in first grade, as was shop and typing (the latter being unusual for that time). Until the middle of the first grade he read English very poorly. After his mother had gone to his school to talk that over with the teachers, she taught him at home. As a result he became, one month later, the best reader in his class.

Also in that first grade—Mitch told me—the teacher once asked who wanted to be absent on the Jewish high holidays. He was puzzled. A friend nudged him to raise his hand. 'There were no Jewish traditions in the house. One Passover my mother cooked a ham. Yet there were rabbis among my mother's ancestry.'

By order of the authorities, Mitch had to move to PS251 after some time, which he found completely boring. 'I just sat there looking out of the window.' He was taken out of the class and put in charge of the school's audio-visual system. When in second grade he helped sixth graders with arithmetic and reading. His teachers were not indifferent to him. 'Some loved me, some hated me.' As to his schoolmates, he never fought with them but rather made them his friends. At age eight he began to lose interest in them. 'I liked parents more than children.' From that time on he had for many years virtually no friends of his age.

In fifth grade his mother taught Mitchell some algebra. He continued to dislike reading. 'I hated libraries and still do.' He did like to read articles on science in the *Encyclopedia Britannica*, however. Understandably he found these largely incomprehensible, and later, when he knew their content, realized that they were probably of no utility to a person of whatever level of understanding. During those years he of course had to take tests, which he passed easily.

When Mitch was about 12 years old he began to develop strong obsessional compulsions about cleanliness—washing his hands all the time. Also, things had to be orderly. When he set his alarm clock he would continue to check whether the lever you had to pull out had not sprung back. This compulsive period came to an end at age 19, for reasons I shall come back to.

At age 12, Mitchell moved to junior high school PS258 in Brooklyn. He was in an SP, a special program, which allowed skipping one year out of three. After one month his algebra teacher sent him away 'because I was always correcting her.' Again he did everything he could not to attend classes. He was once more put in charge of the audio-visual system, a more fancy one this time, and also ran projectors. Furthermore he became secretary of the gym class—which allowed him not to do gym himself—and was on the chess team, 'at which I did medium.' He was also taught French, which he found useless, though later he became more fluent at it as the result of visits to France.

At the end of each year one had to 'take the Regents,' a statewide written examination. Mitch scored 100 out of a 100 in math and science; also high in other subjects.

Also at age 12, Mitch taught himself to play the piano on the instrument at a friend's home. Half a year later his parents bought a piano for his sister. Thereafter Mitchell took some piano lessons, but only six months. He took another half year's lessons when 15 but continued mainly to teach himself and kept playing till at age 19 he left home and no longer had easy access to an instrument. He bought a grand piano after he moved to New York City in 1987. Again he took some exceptionally fine instruction but now only plays infrequently.

The next schooling Mitchell received was at Tilden High School, a good Brooklyn school. He did the three-year curriculum in two and a half. He found the education mostly crummy and the students uninteresting. He was on the math team, which freed him from one school day a week, again worked the audio-visual school systems, and again was gym secretary. His Regents scores were as high as before.

Undergraduate years

Mitch had turned 16 when, in February 1961, he entered the City College of New York, at Convent Avenue and 137th Street in the Bronx. At that time entrance demands were a high-school average higher than 88. Tuition was free except for a $15 admission fee. It was a very good college then, less so later when it was ruled that a high-school diploma sufficed. Mitch traveled between home and CCNY by bus and subway—then costing 15 cents each—which took him one hour 45 minutes each way.

Mitch chose electrical engineering because at about age ten he had found out that these people knew how a radio works—that, as said, had puzzled him greatly as a growing child. Also he had learned that such a degree allowed him to land a job making something like $10 000 a year. It was a five-year program which, however, he completed in three and a half years, taking all the physics and mathematics graduate courses as well. He rapidly realized that it was physics that held the secret of radio waves. He took lots of laboratory courses and went to CCNY summer school to speed things up. His grades were As in subjects in which he was, Cs in those in which he was not, interested. In 1964, at age 19, he obtained the BEE (Bachelor of Electrical Engineering) *magna cum laude*. He missed *summa cum laude* by one thousandth of a point because his lab course grades were only moderate.

Mitch had already taught himself calculus during his last high-school year. That changed his life style of education. Self-instruction became the most important. He had already begun his research as an undergraduate, together with Professor Mansour Javid. His first work dealt with neural networks (then called Adeline) in relation to voice recognition. At that time he also became interested in applications of feedback control to economic problems. In that connection he figured out in 1963 the theory of linear response behavior, a subject that came into general vogue only in 1968.

Graduate education

Mitchell had applied to graduate schools at CalTech, Columbia University, Harvard, MIT, and Princeton, and had been accepted by all. Influenced by experiences of a friend of his brother, he chose MIT, where he started in the summer of 1964. Shortly afterward he became a member of the American Physical Society, of which he is now a Fellow.

As to his living quarters, 'I kept moving,' living in graduate dormitories, then in rooms in Cambridge, Brookline, Belmont. For his six years' stay at MIT—up to and including his PhD—he had financial support from an NSF graduate fellowship for the first three years, then became research assistant to Francis Low. Initially Mitch enrolled in the electrical engineering program, but he knew by now that you need physics to understand how a radio *really* works. Since each student was free to choose his own course schedule, he began concentrating on physics and mathematics. In his first graduate term he applied for a switch to the physics department which was granted the next term.

Since his first term, Mitch had taken courses in quantum and classical mechanics, also in complex functions, in the math department. Of quantum mechanics he recalls not liking the theory of two-body scattering, feeling rather drawn to the idea of studying complex systems. He began to get bored as early as his first term and started, on his own, the study of general relativity theory, reading from cover to cover Landau and Lifshitz's book on that subject, on which he wanted to do his PhD thesis. That turned out to be impossible, however, 'there was no one at MIT during those years to guide that sort of work,' nor anywhere else at that time for that matter. It bothered him that the MIT faculty showed little interest in problems of principle.

During his graduate year Mitch got 100 in all exams, yet received only an overall B mark in his EE course because he would not solve the problems assigned in class. Even before he had finished his graduate studies he was offered an assistant professorship in electrical engineering, which he declined.

As to extracurricular interests: already in his first term he began reading extensively, including philosophy, for example Kant's critique of pure reason, also all of Dostoevski. He also spent several hours a day in the music library, listening to records and reading scores.

One day, in his twentieth year, Mitch drove with some friends to the nearby Lincoln reservoir. While they went for a walk, he strolled by himself to the De Cordova Museum of modern art, which lies on a nearby hill. On the way up he had a revelation. The question came to him: What do people's perceptions, visual, aural, etc. have to do with *the reality* of what they perceive? It came to him that he should know much more than he did about psychology and philosophy. This led him to study the works of Freud, 'the whole stuff.' He also started reading Ernst Mach, Newton's *Principia*, and Galileo's writings. 'I educated myself.' At age 22 he became seriously interested in visual physiology.

I mentioned earlier that, from age eight on, Mitchell had no friends among his contemporaries. So it remained until his last year at CCNY, when he decided he had to do something about that, that he had to make efforts to meet his peers. So he forced himself to go to the cafeteria and strike up conversations, from which he did not derive much inspiration. He did meet several people at this point who were to remain life-long friends, however. His reading of Freud had impressed him very much, but he does not know whether that was of actual help to him in eliminating his obsessional acts. As a result of these readings, self-analysis did become important to him, however.

Another aspect of Mitchell's early years, also mentioned before, was his strongly obsessive behavior. All that vanished when at age 19 he started kissing women. In graduate school he had his first love affair—which in fact was a disaster. At age 23 he started living with a woman. Virtually all his girlfriends—and his two wives—were non-American born.

In 1970 Mitchell received his PhD (the Master's degree is dispensed with at MIT). His thesis advisor was Francis Low, his topic: dispersion relations. This work led to his first publication, jointly with Low.[1] The time had now come for some years of postdoctoral research.

Postdoc positions

Mitchell went first to Cornell for two years, supported half by an instructorship, half by an NSF postdoctoral grant, totalling $10 000 per annum. At that time only 50 such grants were available nationwide. His title was: instructor/research associate. He took teaching seriously, giving courses in variational techniques and advanced nonrelativistic quantum mechanics. He was also partly responsible for a physics course for 500–700 sophomore medical students, in which he managed to include special relativity theory, which, expanded upon, resulted in a paper years later.[2]

During his two years at Cornell, Mitch became completely conversant with everything in theoretical particle physics. He did not find it a domain which illuminated his understanding of the world. Nevertheless he published three more papers on that subject.[3,4,5] Their contents indicate his growing interest in complex systems.

Of the physicists at Cornell, Mitch liked Ed Salpeter and Pete Carruthers. He was impressed with Ken Wilson's skills, liked very much his work on the renormalization group, and was inspired by Ken's lectures on that subject. Mitch respected Bethe's technical abilities but was less impressed with his views on world problems. During his Cornell years, Mitchell met David Finkelstein, from Yeshiva University, whose 'real thinking' about fundamental issues had a strong impact on him.

As Mitch and I were talking about these Cornell experiences the subject naturally turned to his encounters with other physicists. He thinks highly of Steve Weinberg's abilities, mentioning in particular his work on current algebra. He had met Feynman several times but did not have substantial conversations with him until 1981, when he was invited to CalTech and offered a position there. He thinks Feynman and Landau are the last great figures in physics.

After Cornell Mitchell went to the Virginia Polytechnic Institute where Paul Zweifel had obtained funding for one postdoc. He stayed there during 1972–74, his support being again $10 000/year. 'At VPI, Zweifel finished my education on good wines, which had interested me for years and on which I already then knew a lot.' His first job offer, as a graduate student, was as a wine salesman. He has remained an expert on the subject.

Once again Mitchell taught, Banach spaces and C*-algebra among other topics. During his second year he got deeply interested in the nature of time in a discrete universe, inspired by conversations with Finkelstein. He did a lot of work on that subject which, however, has remained unpublished. Also in that year he became professionally interested in the renormalization group.

'These two-year positions made serious work almost impossible. After one year you had to start worrying about where you could go next.'

After Blacksburg, Mitch obtained his first long-term appointment. Carruthers had gone to Los Alamos, where he had become head of the theory department. He offered Mitchell a position there as staff member, with a yearly salary of $22 500 plus funds for travel. Mitch accepted gladly for the science but after considerable soul-searching about the venue, and made his move in 1974.

In 1976, just after recuperating from the ordeal of finishing the most important work of his life—to which I shall turn anon—Mitch met Cornelia Drobowolski at Los Alamos, a German woman who was studying for a Master's degree in German literature. They married in 1978. She brought two young sons, then four and eight, to the marriage. The marriage ended in divorce in 1981, though Mitch has continued to consider both boys as 'his' sons. They remain close and still visit with Mitch in New York.

After the divorce, Mitchell was in dire straits and withdrew from human contacts, most especially in Los Alamos. On the recommendation of friends, he went to consult a Jungian psychiatrist in Santa Fe, whom he saw once a week for eight visits. He found him to be a wise and really humane person who counseled him as well on how to handle practical aspects of his divorce. Mitchell spoke to me with glowing admiration about this man.

In 1986 he married again, with Gunilla Öhman, Swedish born. She is a quite talented artist, both as writer and as painter. My wife and I are happy to count the two of them as dear friends.

Toward chaos[6]

'When I arrived at Los Alamos, Carruthers felt that the time was right, and I was the appropriate person, to see if Ken Wilson's renormalization group ideas could solve the century-old problem of turbulence [a question raised by Wilson himself[7]]. In a nutshell, it couldn't—or so far hasn't—but led me off in wonderful directions.'[8] The wonderful direction was chaos theory, the study of systems that aren't random but appear random. Here and in what follows I shall mean by chaos what is more precisely called dynamical chaos, the apparently random motion of a dynamical system, that is, a system with no random forcing.

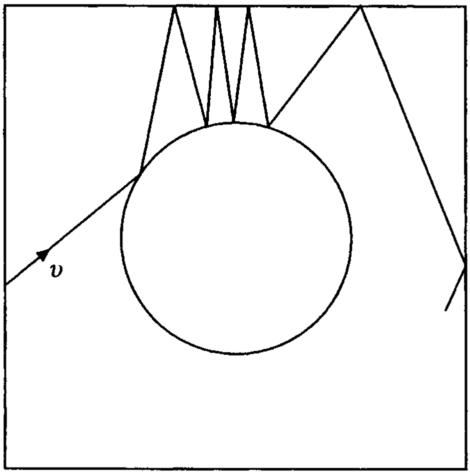
Mitchell's work on this subject started in the 1970s. It is a profound extension of *classical* physics, where by then it had a long history. I sketch a few highlights.

First, turbulence, a term coined by Lord Kelvin[9] derived from the French *tourbillon*, which means whirlpool or eddy. That was a few years following the seminal discoveries of the British physicist Osborne Reynolds,[10] after whom the number is named that marks the transition from regular, laminar, flow to chaotic, turbulent, flow as the flow velocity increases.

It was Henri Poincaré who, also in the late nineteenth century, was the first to realize that the physical laws of motion for a system as simple (pardon the expression) as that consisting of sun, earth, moon cannot be rigorously solved because there are not enough constants of the motion,[11,12] one of the characteristic properties of chaos. Another: in the course of time two orbits of a given system that had started out close together will depart exponentially from each other. Their relative distance grows as $e^{\lambda t}$; the positive number λ is called the Liapunov exponent (also dating from the 1890s[13], named after Aleksandr Liapunov). That was very clear to Poincaré:

> If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still know the situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by the laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible.[14]

Incidentally, in this context use of the *term* chaos to describe this situation dates only from 1975, where it first appears in the title of a paper: 'Period three implies chaos.'[15]



Sinai's billiard.

An elementary example[16] of a chaotic system is Yasha Sinai's billiard: a little sphere moves within a square billiard in which a round obstacle with reflective walls is placed. The motion of the little sphere is deterministic but successive collisions with the obstacle cause neighboring orbits to diverge exponentially. A rough estimate shows that the corresponding Liapunov exponent is given by $\lambda = v/L$, where v is the sphere's velocity and L a representative length.

One last remark on Poincaré's three-body problems. He had discovered that rigorous solutions are impossible but, of course, its orbits can be calculated numerically with high precision—at least for a while. Long-term questions about three-body problems cannot be answered, however. To this day we don't even know whether the solar system is stable; the best present informed numerics indicate that it is *not.* It certainly appears to be so in the short term, but we cannot (yet) give a rigorous answer to the question: can planets escape from the solar system?

Poincaré had no computers. Had those been available to him, he could have tracked three-body orbits for much extended periods of time—while still being unable to answer the escape question till kingdom come.

However that may be, since the 1960s computers have played an absolutely crucial role in advancing the understanding of chaotic phenomena.

One day in the winter of 1961 the research meteorologist Edward Norton Lorenz was in his office at MIT, busy as usual mapping weather on his Royal McBee LGP-30 computer—a quite clumsy one by modern standards. He would print out one parameter or another, the direction of an air stream, air pressure, temperature, and so on. It was 'toy weather,' that is, he used simple, simplified, purely deterministic equations (now known as the Lorenz equations) which mimic the earth's real weather. On that day he wanted to check an earlier result, so he once again typed in the initial conditions, then went out for a cup of coffee. When he returned he saw something quite unexpected. In his own words:

> During our computations we decided to examine one of the solutions in greater detail, and we chose some intermediate conditions which had been typed out by the computer and typed them in as new initial conditions. Upon returning to the computer an hour later, after it had simulated about two months of 'weather,' we found that it completely disagreed with the earlier solution. At first we expected machine trouble, which was not unusual, but we soon realized that the two solutions did not originate from identical conditions. The computations had been carried internally to about six decimal places, but the typed output contained only three, so that the new initial conditions consisted of old conditions plus small perturbations. These perturbations were amplifying quasi-exponentially, doubling in about four simulated days, so that after two months the solutions were going their separate ways.[17]

Here Lorenz had made quantitative[18] the sensitive dependence on initial conditions, which the ever-prescient Poincaré had noted in the paragraph immediately following my previous quotation:

> Why have meteorologists such difficulty in predicting the weather with any certainty? . . . We see that great disturbances are generally produced in regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but exactly where they are not in a position to say; a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared . . . Here, again, we find the same contrast between a very trifling cause that is inappreciable to the observer, and considerable effects, that are sometimes terrible disasters.[14]

This situation is now popularly known as the butterfly effect: the trifling event of a butterfly fluttering in the air over Kyoto can lead to a disastrous storm over Chicago. Hence, in Lorenz's words: '*Precise very-long-range [weather]* forecasting would seem to be non-existent'[18] (my italics).

'Years later, physicists would give wistful looks when they talked about Lorenz's paper[18]—that beautiful marvel of a paper . . . All the richness of chaos was there . . . By then it was talked about as if it were an ancient scroll, preserving secrets of eternity.'[19] I respect most of all Lorenz's realization that what he had seen was not machine trouble but something profoundly new. There is more novelty in that paper; for example it contains the first picture[20] of an infinitely tangled abstraction now called a strange attractor.[21]

Lorenz's paper[19] marks the beginning of a new era in science, the quantitative study of chaos, resented by some, evangelically embraced by many. It has been written that 'chaos presages the future as none will gainsay.'[22] Intuition in chaos continues to be developed with the help of computers, mathematics has become its experimental tool par excellence. 'The heart of chaos is mathematically accessible.'[23]

By now chaos is ubiquitous. After its start in meteorology one employs chaos theory in studies of turbulence, astronomy and cosmology, laser optics, acoustics, plasma physics, accelerator physics, chemical reactions. Also in animal populations, epidemiology (chaos was found in records of New York's measles epidemics,[24] generally the fraction of a population infected at a given time), the theory of learning (the bits of information that can be remembered after a given period of time), the propagation of rumors (the number of people who have heard a certain rumor after some period of time), the flow of car traffic, the relationship between commodity quantity and price on the stock market.[25] All of which has led to partnerships between physics and other disciplines. Literature on chaos has exploded. A list of publications on that subject published between 1963 and 1983 contains about 1000 items; in the later 1980s more than 2000 further background papers appeared.[26] In 1977 the first conference on chaos was held. Such meetings as well as journals dealing exclusively with chaos now abound.

One last item needs to be mentioned before I turn to Feigenbaum's contributions to the theory of chaos: bifurcations, once again first mentioned by the great Poincaré[27] in lectures given at the Sorbonne in 1900, in the course of describing motions of cylindrical columns of liquids. At that time he used the expression *échange des stabilités* for what now is called bifurcations. It is astonishing to me that, to my knowledge, Poincaré is never mentioned in literature on bifurcation.

The modern era of popular awareness of physical and biological studies in bifurcation began in the 1970s, with the paper 'Simple mathematical models with very complicated dynamics'[28] by Sir Robert McCredie May, then a professor of biology in Princeton, later that university's dean of research, now the chief scientific advisor to the British government. The paper contains 'An evangelical plea . . . not only in research, but also in the everyday world of politics and economics, we would be better off if more people realized . . . the wild things that simple non-linear equations can do.' (As we shall see, that problem was already solved when May wrote his article.)

May's specialty was population biology, particularly the ecological problem of how populations behave over time. That question goes back at least to Thomas Malthus, who had posited a scenario in which populations show unrestrained growth, so that, he feared, they would outrun the food supply. In mathematical terms: let x_t denote a population at time t, and x_{t+1} the same one year later. Then, Malthus assumed, $x_{t+1} = rx_t$, where r represents the rate of growth.

The equation May analyzed was

$$x_{t+1} = f(x_t) \qquad (1)$$
$$f(x_t) = rx_t\,(1-x_t),$$

known as the non-linear logistic difference equation. In the jargon of non-linear dynamics $f(x_t)$ is called a map. Here 'population' is treated as a fraction between zero and one. Zero represents extinction, one the maximum population. That equation had been studied well before, but May was the first to appreciate how very rich in information that innocent-looking equation is.

The reader may like to have a simple pocket calculator at hand in order to follow what happens as the growth parameter rises:

For $r < 1$, populations are attracted to zero. Example: $x_1 = 0.4$, $r = 0.5$: $x_2 = 0.12$, $x_3 = 0.053, \ldots$

For $r > 4$, all populations tend to minus infinity. Example: $x_1 = .4$, $r = 5$: $x_2 = 1.2$, $x_3 = -1.2$, $x_4 = -13.2, \ldots$

Populations behave in a less trivial way when r lies between 1 and 3. Example: $x_1 = 0.02$, $r = 2.7$. The population wobbles up and down, settling finally at the value 0.6296.

The real excitement starts when r lies between 3 and 4. Example: $x_1 = 0.4$, $r = 3.1$. The population sequence in the first eight years is:

0.4	0.744
0.590	0.770
0.549	0.777
0.539	0.770,

to be read as follows: across the first row, then the second, etc. Thus the population is 0.4 in the first year, 0.744 in the second, 0.590 in the third. Bifurcation has set in! The population oscillates between the two values in alternating years. Different values for x_1 converge to the same two-year cycle.

Now start again with $x_1 = 0.4$ but raise r to 3.5 You will find the sequence:

0.4000,	0.8400,	0.4704,	0.8719,
0.3908,	0.8332,	0.4862,	0.8743,
0.3846,	0.8284,	0.4976,	0.8750,
0.3829,	0.8270,	0.4976,	0.8750,
0.3829,	0.8270,	0.5008,	0.8750,
0.3828,	0.8269,	0.5009,	0.8750,
0.3828,	0.8269,	0.5009,	0.8750, etc.,

again to be read in the order first row, second row, etc. We see bifurcation of bifurcation! The two-year cycle has become a four-year cycle. Raising r further yields 8, 16, 32, . . . bifurcations. The range of r values wherein any number of bifurcations occurs diminishes progressively as that number increases until one reaches the accumulation value $r = 3.5699 \ldots$, where (as was theoretically proved later) the period has doubled *ad infinitum*. For higher r, the behavior becomes erratic, aperiodic. We have reached the chaos regime.

In 1973 it was conjectured[29] that the behavior described for the logistic difference-equation holds *qualitatively* for all $f(x_t)$ which have a maximum (at $x_t = 0.5$ in the logistic case) and fall off monotonically on both sides.

Within the region of chaos one finds an infinite number of ever smaller r-ranges for which the system becomes again periodic.[30] The behavior of this logistic equation, so elementary looking at first sight, is utterly remarkable. Go figure.

The Feigenbaum numbers

At Los Alamos it soon became clear to his fellow scientists that Mitchell was a very smart man worth talking to—he himself always talked rapidly—when they got stuck on their own work. They knew that he was in deep thought nearly all the time but did not produce any scientific papers. The main problems he was brooding about concerned big, complex, systems, how physics sufficed (if it did) to describe their reality.

By August 1975—Mitch has told me—he had his first new result, obtained by using his first programmable calculator, an original HP65, given him in December 1974 as a perk for having been promoted to staff member at Los Alamos. This is what he found that August, after a round-about path of analytical thought. Let r_i be the parameter value where the i-th bifurcation sets in, and let $\Delta_i = r_{i+1} - r_i$. Then the Δ_i sequence *asymptotically* converges *geometrically*[31] with increasing i.

> By $r_i = 4$ it was already clear that Δ_i was converging geometrically. One sees this by noticing that the difference of successive values decreases by a constant ratio, which quickly appeared to be about 5. By $i = 7$ the next term in the series already solved the equation to machine precision, which was exceeded beyond $i = 8$. But the ratio of differences itself was converging, down to 4.669 before precision deteriorated . . . Now this was curious and extraordinary . . . What in this florid calculation was providing geometrical convergence? . . . I was so struck by this that I spent part of the day trying to see if 4.669 was close to various simple combinations of numbers and so forth. Nothing at all clear turned up . . .
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> I spent the first week of October visiting CalTech . . . [when] I suddenly was taken aback by a memory. Stein had told me that the doubling was the same for anything that looked like a bump. In the MSS paper[29] I had looked over almost a year earlier, I suddenly recalled that $x_{t+1} = r \sin \pi x_t$ showed identical behavior to Eq. (1). The day I returned home I decided immediately to check if sin x actually

doubled. Indeed it did, but at 1 second per trigonometry the wait was painful. I recalled there was an easy way to guess the next value, and by $n = 4$ again realized there was geometric convergence. By my efforts to fit the ratio, the new result settling down to 4.662 smelled familiar. A quick rummage through my drawer resurrected the sheet with 4.669 for the [logistic difference equation].

Without an instant's hesitation I experienced an overwhelming excitement that I had stumbled upon a piece of the godhead.

I immediately called Stein. No, he didn't know that the doublings converged geometrically and was deeply skeptical that a universal *quantitative* entity could exist. I went over to his office to show him the numbers which had him respond with a repressed anger that I had no right to such suppositions based on just three identical figures. But *twelve* figures would convince him.

Nevertheless, I called my parents that evening [on October 22] and told them that I had discovered something truly remarkable, that, when I had understood it, would make me a famous man.

A colleague, one of the most knowledgeable computer users, gave me a FORTRAN instruction list book to look at, and told me next morning he'd help me onto a serious Los Alamos computer. His few hours of instruction on the system, editor and the easiest way to get output were extraordinary. I had, under my own steam, 4.6692 by the end of the day. This wasn't my limitation. Rather, for naive iteration, 1/3 of the precision of the machine is all one can do. So the next day . . . another computation expert gave me crucial pointers on how to use 29 digit CDC double precision arithmetic. Finally, the next day, some four days after the last encounter, I marched into Stein's office with 4.66920160 . . . agreed to 11 figures for four different problems. This time he concurred, and took out his 'dictionary' of numbers—an ordered list of several hundred pages of decimals and what they represent. By the '9' nothing was close.

So far I have told about how a number I called δ was born. With this as really the sole clue, I knew it foreshadowed an entire world.[32]

Up to twelve decimals, his result was

$$\frac{\Delta_i}{\Delta_{i+1}} \rightarrow \delta = 4.669201660910 \text{ as } i \rightarrow \infty.$$

Mitchell made his next step into this new world in early 1976 when he conceived the crucial idea that the theory was to be expressed in terms of functional equations, and that not just δ but *all* the chaos dynamics must be quantitatively universal. 'This was the great jewel.'[33] The first outcome of this line of thought dealt with the magnitudes ε_i of the separation between the two arms of the i-th bifurcation about the maximum. He found that, asymptotically, this separation is reduced by a constant value a when going from one doubling to the next:

$$\frac{\varepsilon_i}{\varepsilon_{i+1}} \rightarrow a \text{ as } i \rightarrow \infty.$$

α is 'another universal constant and this one for the actual dynamics!'[34]

'After an extraordinary amount of analytical computer effort, some two and a half months, every day, 22 hours a day, until I required medical attention in mid-March'[34]—he had lived practically on coffee and cigarettes only—he had his functional equation, which 'justified the big dream that dynamics . . . when appropriately complex in behavior, knew how to perform independently of details.'[34] He had completed the search for his 'universal function,' an achievement he has called 'the most extraordinary discovery I have made in my life.'[34]

The analytic part of the work consisted in finding the functional equation for the universal function; the numerical portion in evaluating α which resulted in (again up to 12 decimals)

$$\alpha = 2.502907875095$$

The numerics to find the first three decimals had been performed with his HP65. Thereafter he had to turn to 'wholesale powerful computation.'[34]

When done, the doctor prescribed a modest amount of valium and an enforced vacation.

δ, the period doubling parameter, and α, the separation parameter, are now known as the Feigenbaum numbers. To the best of my knowledge these are the only dimensionless universal numbers named after a person in the twentieth century. Mitchell has calculated their first 100 decimals, others may have gone further. It is not known whether these numbers are transcendental. It would be highly surprising if they were not.

Mitchell's universal function is not known in closed analytic form. (It is analytic over some domain.) All its consequences are obtained by a combination of analytics and significant amounts of numerical calculation. In a paper[35] published in 1992(!), we read: 'Some of us have been wondering for a long time what the *domain of validity of these discoveries* is and what *techniques from dynamics* need to be employed or invented to make a proof' (author's italics). These questions are answered in that paper[35] with the help of newly invented methods.

I shall not try to sketch the contents of that derivation and proof of the universal function because, frankly, I could not follow the arguments myself! Even Mitchell calls that paper 'exceptionally heavy-duty mathematics.' Instead I refer to papers by Feigenbaum; the first,[36] heuristic one, completed in April 1976, was published in 1978, updated in an appended afterword which acknowledged a seminal contribution made by Predrag Cvitanovié in May 1976, crucial to the technical paper[37] published in 1979. A letter written in 1979,[38] followed by a full version in 1980,[39] enlarged this work to an arbitrary number of dimensions, thus making the first contact with the real world. He has also published[40] a semi-popular version of this work.

It has been Mitch's good fortune that he had begun this work with the help of a pocket calculator. That had given him time to reflect and to guess ahead.

> I know no one had discovered δ prior to myself . . . So far as I can tell, had I not had a training that made me eschew computers, had I not so thoroughly enjoyed the computation and 'meaning' of numbers . . . and had the HP65 not been so excruciatingly slow, I wouldn't have discovered δ either. It is a *sine qua non* of emergent behavior. But how do you see it if you don't know what it looks like? After all, fate and luck play all too significant a role.[41]

End notes

1. Feigenbaum's two papers[36,37] were both rejected by journals, the first after a half-year delay. Mitchell still keeps the rejection letters in a desk drawer. 'Every novel paper of mine, without exception, has been rejected by the refereeing process. The reader can easily garner that I regard this entire process as a false guardian and wastefully dishonest.'[42] By 1977 over a thousand preprints of the first paper had been shipped, however.

   Mitch lectured far and wide on his work, starting in May 1976, in Princeton, in August at a Gordon conference, in September before his first international audience, at Los Alamos. In 1981 he spoke at CalTech.

   > That colloquium proved to be the most enjoyable and electric one in my career: it rapidly developed into a dialogue between Feynman in the front row and myself. After the talk I went up to his office. 'You know, I'm envious of you,' he said. 'Come on, you of all people can't be envious of *me*.' 'Well, maybe you're right,' he rejoined.[42]

2. During and after 1979, Mitchell continued to publish on chaos theory.[43] Among his other work I note in particular that on the construction of geographical maps. This has led to new improved editions of the *Hammond Atlas*. In the introductory notes[44] to this volume we read:

   > Using fractal geometry to describe natural forms such as coastlines, mathematical physicist Mitchell Feigenbaum developed software capable of reconfiguring coastlines, borders, and mountain ranges to fit a multitude of map scales and projections . . . Dr. Feigenbaum also created a new computerized type placement program which places thousands of map labels in minutes, a task which previously required days of tedious labor.

   Mitchell has written two papers[45] on the mathematics of map making.

3. The previous sketches about chaos refer exclusively to one-dimensional problems. Much important work on chaos in more dimensions and on rigorous mathematics

has meanwhile been performed by him[39] and by others. I refer to refs 26 and 49 for reviews and collections of major reprinted papers on these various topics, and take this opportunity to express my respects and apologies for not going into detail on the work of their authors.

4. In 1982 Feigenbaum left Los Alamos to assume a professorship at Cornell, until 1986. Thereafter he became professor in the Rockefeller University, where he was named the first Toyota professor, a chair endowed by the Toyota Motor Corporation. In 1984 he received a MacArthur Foundation award.

5. In 1979 Albert Joseph Libchaber (born in Paris, still a French citizen), then working at the École Normale Supérieure, and assisted by Jean Maurer, an engineer, published[47] his first preliminary results of observations on bifurcation cascades. The experimental arrangement is astoundingly small, elegant, and simple. A five cubic millimeter rectangular convective cell is filled with liquid helium. The temperature is made to vary from 2.5 to 4.5 degrees Kelvin, the pressure from 1 to 5 atmospheres. When the fluid is heated from below, at low heating rates, no flow occurs. At higher rates, time-independent convection is set up. At yet higher rates, a periodic time-dependence occurs. At still higher rates they first observed a series of successive period doublings, followed by a very chaotic regime with a broad band spectrum.

   In this paper[47] there is no mention of Feigenbaum's work. In a more elaborate paper[48] with the appropriate title 'Helium in a small box,' published in 1982, one finds a section, however, entitled 'The period doubling bifurcation to chaos, Feigenbaum scheme,' in which it is noted 'The qualitative picture proposed by Feigenbaum seems to be correct. Quantitatively there are discrepancies which may be associated to the fact that we observe only the very first bifurcations.' Later with more extensive experiments, this discrepancy disappeared. The questions arise: when did the two men know of each other's work and how did they know it? To find out, I went to consult both Albert, also a friend of mine, and Mitchell. This is what I found out.

   Albert has told me that, when he began this work, he did not know of Mitchell's theoretical considerations. Then why did he choose this problem? 'It was in the air,' is all he said to me. Mitchell already knew of Albert's data in 1979, as is seen from his Letter[38] which came out in December of that year, where he quoted the first paper of ref. 47 and stated 'We find excellent agreement with the recent experimental data of Libchaber and Maurer.' The two met for the first time soon afterward, in Paris. Feigenbaum has written, '1979 was a banner year . . . Dynamical systems only became "science" after Libchaber's measurements in the summer of 1979 showed that a fluid can make a transition to turbulence via period doubling with the generic values of α and δ.'[42]

   Since 1983, Libchaber has held professorships in Chicago, Princeton, and currently in the Rockefeller University. He has received wide recognition, including the French Legion of Honor. He is married and has three talented sons.

6. In 1986 Feigenbaum and Libchaber were in Jerusalem to share the Wolf Prize in physics ($100 000). Mitchell was cited 'for his pioneering theoretical studies

demonstrating the universal character of non-linear systems, which has made possible the systematic study of chaos,' Albert 'for his brilliant experimental demonstration of the transition to turbulence and chaos in dynamical systems.'

In the accompanying press release it is further noted: 'The impact of Feigenbaum's discoveries has been phenomenal. It has spanned new fields of theoretical and 'experimental' mathematics . . . It is hard to think of any other development in recent theoretical science that has had so broad an impact over so wide a range of fields, spanning both the very pure and the very applied.'

7. I regard chaos theory as one of the great revolutions in twentieth-century physics, along with relativity and quantum mechanics. No two of these are alike, of course. In particular chaos has not produced a paradigm shift (if I understand correctly what that peculiar concept means). One physicist has put it well: 'Relativity eliminated the Newtonian illusion of absolute space and time; quantum theory eliminated the Newtonian dream of a controllable measurement process; and chaos eliminates the Laplacian fantasy of deterministic predictability.'[49]

There are some physicists, famous, yes, wise, no, who have stated in print that the end, the completion, of physical theory is in sight. I am not of that persuasion. 'Twenty years ago, no physicist knew of chaos and, what is more important, of its prevalence.'[50] For centuries most of theoretical studies in physics had been focussed on linear systems or on the linear approximations to more realistic situations. It is only in the last 20 years or so that methods have been found to handle some of the mathematical intricacies which non-linearity brings to a realistic description of natural phenomena.

Mitchell and I believe that we know much, yet little indeed. We are both convinced that the truly astonishing developments described in this essay are but early beginnings, that more surprises are to come. When? From where? In other areas too? Who can tell . . .

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