Overview of Topics

① Laplace Transforms & PDEs

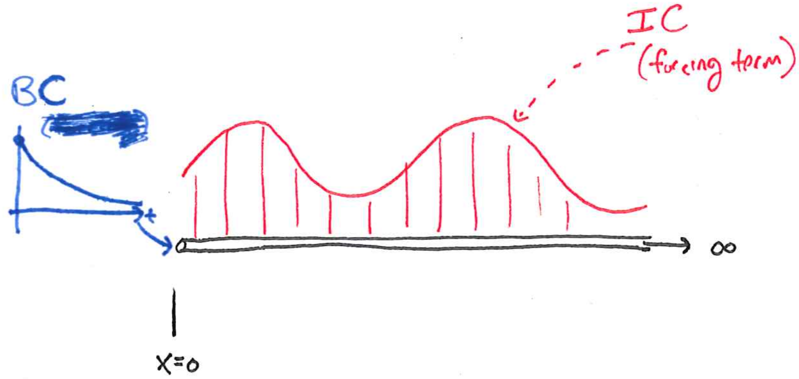
(*) BCs and ICs

② Summary of Solution Methods for PDEs

Fourier Transforms are good for solving PDEs on an infinite domain (with decaying BCs at ∞) with ICs.

Laplace Transforms are good for PDEs on semi-infinite domains with ICs and a BC.

Example: Heat Eq$^{\underline{n}}$



$$u_t = u_{xx} \quad \text{on } 0 \leq x < \infty$$

IC $\quad u(x,0) = \sin(4x) \quad$ at $t=0$

BC $\quad u(0,t) = e^{-t} \quad$ at $x=0$

Solve for $u(x,t)$ on $0 \leq x < \infty$.

$\mathcal{L}\{\cdot\}$ with respect to time, because we have a single derivative in time, and a single IC at $t=0$...

$$s\bar{u}(x,s) - \underbrace{u(x,0)}_{\text{forcing term}} = \bar{u}_{xx}(x,s)$$

ODE in x (2nd order)

Because $u(x,0) \neq 0$, we have a general (homogeneous) solution and a particular solution:

$$\bar{u}(x,s) = \bar{A}(s)e^{-\sqrt{s}\,x} + \underbrace{\bar{B}(s)e^{\sqrt{s}\,x}}_{\substack{B=0 \text{ for bounded} \\ \text{solution}}} + \underbrace{\bar{\Psi}(x,s)}_{\text{Particular sol}^{\underline{n}}}$$

Lets find $\bar{\Psi}(x,s)$.. Assume $\bar{\Psi}(x,s) = \bar{C}(s)\cos(4x) + \bar{D}(s)\sin(4x)$

$$\underbrace{s\bar{C}(s)\cos(4x) + s\bar{D}(s)\sin(4x)}_{s\bar{\Psi}} - \underline{\sin(4x)} = \underbrace{-16\bar{C}(s)\cos(4x) - 16\bar{D}(s)\sin(4x)}_{\bar{\Psi}_{xx}}$$

$$\Longrightarrow \left[s\bar{C}(s) + 16\,C(s)\right]\cos(4x) + \left[s\bar{D}(s) + 16\,\bar{D}(s) - 1\right]\sin(4x) = 0$$

$$\Longrightarrow \quad \bar{C}(s) = 0, \qquad \bar{D}(s) = \frac{1}{s+16}$$

coefficients of particular solution $\bar{\Psi}$

(Continued)

Now, we can determine $\bar{A}(s)$ by setting $x=0$ and using Boundary Condition $u(0,t) = e^{-t}$.

$$\bar{u}(0,s) = A(s)e^0 + \frac{1}{s+16}\sin(0) = A(s) \implies A(s) = \bar{u}(0,s) = \mathcal{L}\{e^{-t}\}$$

$$\implies A(s) = \frac{1}{s+1}$$

Solution in frequency domain:

$$\bar{u}(x,s) = \frac{1}{s+1} e^{-\sqrt{s}\,x} + \frac{1}{s+16}\sin(4x)$$

In time domain:

$$u(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{e^{-\sqrt{s}\,x}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+16}\right\}\sin(4x)$$

$$u(x,t) = \left[e^{-t}\right] * \left[\frac{1}{2\sqrt{\pi t^3}} e^{-\frac{1}{4t}}\right] + e^{-16t}\sin(4x)$$

Question

Is it possible to solve using $\mathcal{L}\{\cdot\}$ w.r.t. space?

No, because I would need two BCs since two 'x' derivatives

$$\mathcal{L}\{u_{xx}\} \text{ wrt 'x'} = s^2\bar{u}(s,t) - s\,u_x(0,t) - u(0,t)$$

we don't have this info...

Physical: rope or string being shaken at the left end... creates traveling waves

$u_{tt} = c^2 u_{xx}$

$u(0,t) = f(t)$ - BC

$u(x,0) = 0$
$u_t(x,0) = 0$] - zero ICs are good for $\mathcal{L}\{\cdot\}$

creates travelling wave solutions!

Take $\mathcal{L}\{\cdot\}$ w.r.t. time since I have two time derivatives and two ICs, . $u(x,0)$ and $u_t(x,0)$:

$$s^2 \bar{u}(x,s) \underbrace{- s u_t(x,0) - u(x,0)}_{=0 \text{ (Good!)}} = c^2 \bar{u}_{xx}$$

$$s^2 \bar{u} = c^2 \bar{u}_{xx}$$ ODE in 'x'

$$\bar{u}(x,s) = \bar{A}(s) e^{-(s/c)x} + \underbrace{\bar{B}(s) e^{+(s/c)x}}_{=0 \text{ for bounded sol}^{n}} = \bar{A}(s) e^{-(s/c)x}$$

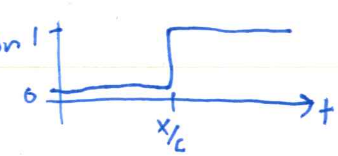
$$\bar{u}(0,s) = \bar{f}(s) = \bar{A}(s) e^{0} \Rightarrow \bar{A}(s) = \bar{f}(s)$$

Solution:

$$\boxed{\bar{u}(x,s) = \bar{f}(s) e^{-(s/c)x}}$$ frequency

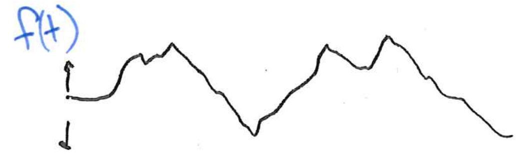
$$\boxed{u(x,t) = H(t - \tfrac{x}{c}) f(t - \tfrac{x}{c})}$$ time

Heaviside function!



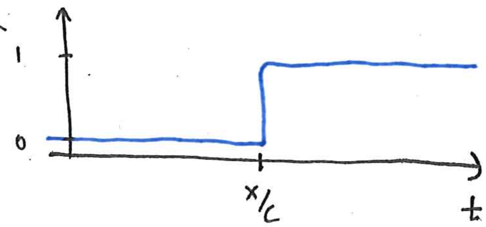
(continued)

$$u(x,t) = H\left(t-\frac{x}{c}\right) f\left(t-\frac{x}{c}\right)$$

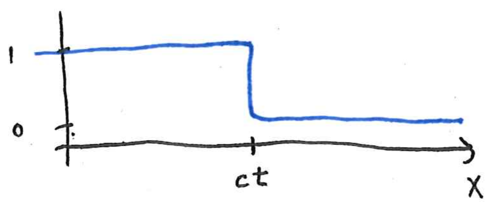


f(t) is shaking the string at one end

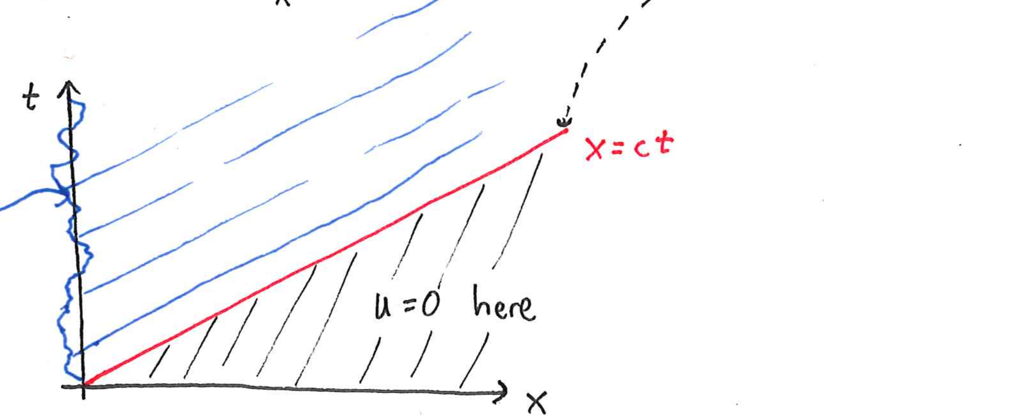
fix x and vary t



fix t and vary x



or... vary both



This is called an x-t diagram, and it is very useful for capturing the physics of waves (i.e. traveling waves, acoustic waves, shock waves, ...) that arise in hyperbolic PDEs (i.e. $u_{tt}=c^2u_{xx}$ or $u_t=cu_x$, ... etc.)

Wave Equation (Hyperbolic)

$u_{tt} = c^2 \nabla^2 u$ $\qquad$ $u_{tt} = c^2 u_{xx}$

Heat Equation (Parabolic)

$u_t = \alpha^2 \nabla^2 u$ $\qquad$ $u_t = \alpha^2 u_{xx}$

Laplace's Equation (Eliptic)

Separation of Variables: $u(x,t) = X(x)\,T(t)$

or $u(x,y) = X(x)\,Y(y)$

Fourier Transform: Good for infinite domains with ICs

Laplace Transform: Good for semi-infinite domains with ICs and BCs.

Homogeneous 2nd Order Linear PDEs:

(A) Wave Equation: $u_{tt} = c^2 \nabla^2 u$ $\quad u_{tt} = c^2 u_{xx}$ (hyperbolic)

(B) Heat Equation: $u_t = \alpha^2 \nabla^2 u$ $\quad u_t = \alpha^2 u_{xx}$ (parabolic)

(C) Laplace's Equation: $\nabla^2 u = 0$ $\quad u_{xx} + u_{yy} = 0$ (elliptic)

Cauchy Problem: Case (A): $u(x, t=0) = u_0(x)$; $u_t(x, t=0) = u_1(x)$

case (B): $u(x, t=0) = u_0(x)$

Proposition: Solutions of $u_t = \alpha^2 u_{xx}$; $u(x,0) = u_0(x)$ (B)

with $u(0,t) = 0$, $u(1,t) = 0$ [Dirichlet]

or $u_x(0,t) = 0$, $u_x(1,t) = 0$ [Neumann]

or [mixed] BC's are unique!

Proposition: Solutions of $u_{tt} = c^2 u_{xx}$; $u(x,0) = u_0$; $u_t(x,0) = u_1$ (A)

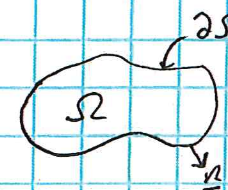
with $u(0,t) = 0$ or $u_x(0,t) = 0$

and $u(1,t) = 0$ or $u_x(1,t) = 0$ are unique.

Laplace & Poisson Equations: (C)

$$\begin{cases} \Delta u = 0 \\ \Delta u = f(x) \end{cases} \quad \text{on } \Omega \subseteq \mathbb{R}^2 (\mathbb{R}^3)$$

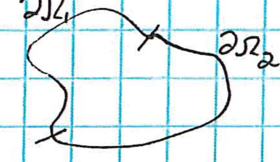
$u|_{\partial\Omega} = g(s)$ [Dirichlet]



$\frac{\partial u}{\partial n} = \nabla u \cdot n = g(s)$ [Neumann]

$\frac{\partial u}{\partial n} + h(s) u = g(s)$ [Robin]

$u = g_1$ on $\partial\Omega_1$, $\frac{\partial u}{\partial n} = g_2$ on $\partial\Omega_2$ [Mixed]



Proposition I: ∃ at most one sol^n to Dirichlet, Robin & Mixed problems. (maybe none but if ∃ then!).

oposition II: Any 2 sol^{ns} of Neumann problem differ by a constant.