L24: March 2, 2015 ME565, Winter 2015

Overview of Topics

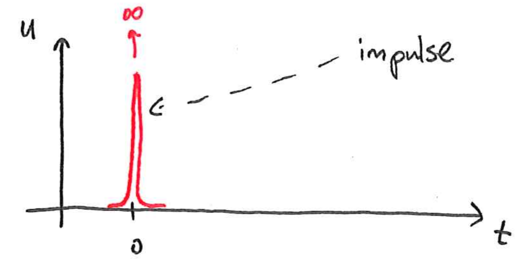
1. Convolution integrals, forcing, and the impulse response

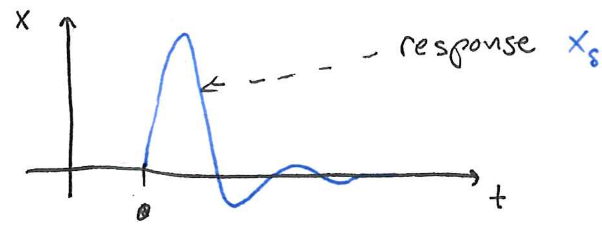
2. Linear systems and the connection between "impulse response", "frequency response" and ODEs

3. Systems of ODEs

Impulse Response ...

Say $u(t) = \delta(t)$





For a given ODE, we will have a transfer function $G(s)$.

$\bar{u}(s) = 1$
for $u(t) = \delta$

$$\left.\begin{aligned} \bar{X}(s) &= G(s)\,\bar{u}(s) \\ \bar{X}_\delta(s) &= G(s) \cdot 1 \end{aligned}\right\} \Longrightarrow x_\delta(t) = \mathcal{L}^{-1}\{G(s)\}$$

Impulse response $x_\delta(t)$ is the inverse Laplace transform of the transfer function!!

Or, we could work exclusively in time domain ...

Example: $\dot{x} - \lambda x = u(t) = \delta(t)$ (assume zero ICs)

Integrate both sides: $\displaystyle\int_{-\varepsilon}^{\varepsilon} \dot{x}(t)\,dt - \lambda \int_{-\varepsilon}^{\varepsilon} x\,dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt$

$$x(\varepsilon) - x(-\varepsilon) = 1$$

If $x(0) = 0$, then $x(0^+) = 1$ ...

Impulse response is like starting the system off with an initial condition.

Matlab: >> impulse (sys)

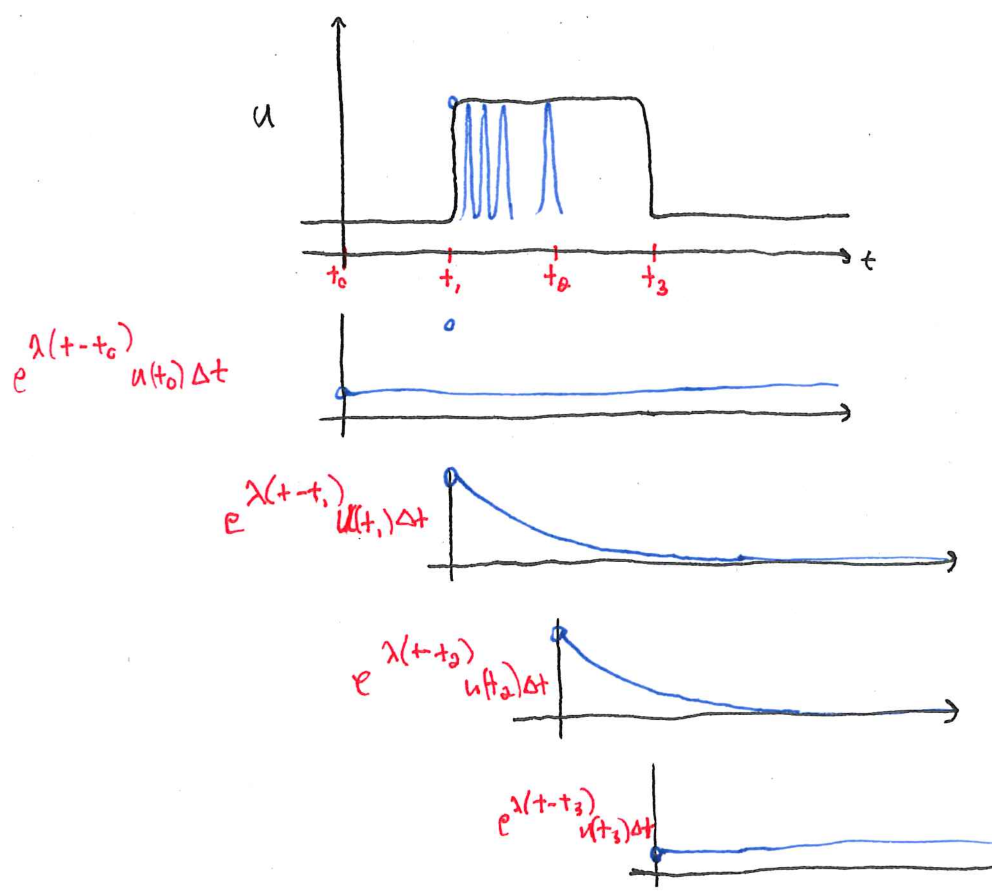
Simple Example of Convolution:

$\dot{x} = \lambda x + u \,, \quad x(0) = 0$

$u(t) =$ 

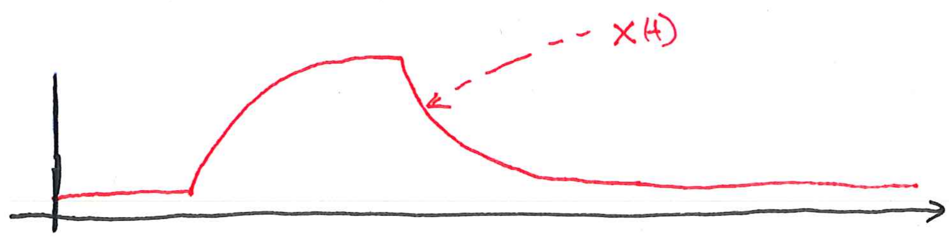
$$x(t) = \int_{-\infty}^{\infty} e^{\lambda(t-\tau)} u(\tau)\, d\tau$$

$$= \int_{0}^{\infty} e^{\lambda(t-\tau)} u(\tau)\, d\tau \quad \text{if } u(\tau) = 0 \text{ for } \tau < 0.$$



We may think of $u(t)$ as a continuous train of little infinitesimal area delta functions (with area equal to $u(t)dt$).

Each of these tiny "impulses" starts an impulse response at time τ, and we add all time shifted impulses up to get $x(t)$.



```
clear all, close all, clc
s = tf('s');
lambda = 1;
sys = 1/(s+lambda);

dt = 0.1;
T = 15;
t = dt:dt:T;
u = 0*t;
u(end/3:2*end/3) = 1;
% u = sin(t);
[y,t] = lsim(sys,u,t);
[yimp,timp] = impulse(sys,t);

tsol = dt:dt:2*T;
xsol = 0*tsol;
figure
for k=1:length(t)
    subplot(3,1,2)
    plot(timp+t(k),yimp,'k');
    axis([0 30 -.1 1.1]);
    subplot(3,1,1)
    plot(t,u,'k');
    hold on
    plot(t(k),u(k),'ro');
    axis([0 30 -.1 1.1]);

    xsol(k:k-1+length(t)) = xsol(k:k-1+length(t)) + dt*u(k)*yimp';
    subplot(3,1,3)
    plot(tsol,xsol,'k');
    axis([0 30 -.1 1.1])
    drawnow
    pause(0.05)
end
```

```
clear all, close all, clc

s = tf('s');
sys = 1/(s^2+5*s+4);

figure
bode(sys)

t = 0.01:.01:100;
u = sin(t);
y = lsim(sys,u,t);
figure
plot(t,y,'k')
%%
w = 1;
s = i*w;
sys = 1/(s^2+5*s+4);
mag = abs(sys)
phase = angle(sys)

hold on
plot(t,mag*sin(t+phase),'r--')
```

More generally, consider a system of ODEs with input:

input(s) forcing

$$\overbrace{\dot{x} = A x}^{\text{dynamics}} + B u$$

$$y = C x$$

internal system state

output measurement(s)

$$\mathcal{L}\{\cdot\} \Rightarrow \quad s\bar{x}(s) - x(0) = A\bar{x}(s) + B\bar{u}(s)$$

$$\bar{y}(s) = C\bar{x}(s)$$

$$(s\mathbb{I} - A)\,\bar{x}(s) = x(0) + B\bar{u}(s)$$

if $x(0) = 0$ ... for simplicity

$$\bar{x}(s) = (s\mathbb{I} - A)^{-1} B\bar{u}(s)$$

$$\bar{y}(s) = C\bar{x}(s) = C(s\mathbb{I} - A)^{-1} B\bar{u}(s)$$

$$\Rightarrow \boxed{\frac{\bar{y}(s)}{\bar{u}(s)} = C(s\mathbb{I} - A)^{-1} B}$$

Input-Output Transfer function $G(s)$.

Spring-mass-damper

$$\left.\begin{bmatrix} \dot{x} = v \\ \dot{v} = -\omega_0^2 x - \omega_0 \xi v + u(t) \end{bmatrix}\right\} \Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\xi\omega_0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

Measure position 'x'

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{0}_{D} \cdot u(t)$$

Matlab: `>> sys = ss(A,B,C,0);`

Convolution, transfer functions, and the impulse response (Green's functions)

ODE:
Time Domain

Given an ODE (w/forcing) $\quad \dot{\underline{x}} = A\underline{x} + Bu \quad ; \quad \underline{x}(0) = \underline{0}$

-or- $\quad \ddot{x} + 3\dot{x} + 2x = u \quad ; \quad x(0)=0, \; \dot{x}(0)=0$

Frequency Domain:
Transfer Function

It is possible to write in the frequency domain:

$$\bar{\underline{x}}(s) = (s\mathbb{I} - A)^{-1} B \bar{u}(s)$$

-or-

$$\bar{x}(s) = \frac{1}{s^2+3s+2}\bar{u}(s)$$

In terms of the transfer function $G(s) = (s\mathbb{I}-A)^{-1}B$

or $\quad G(s) = \dfrac{1}{s^2+3s+2}$

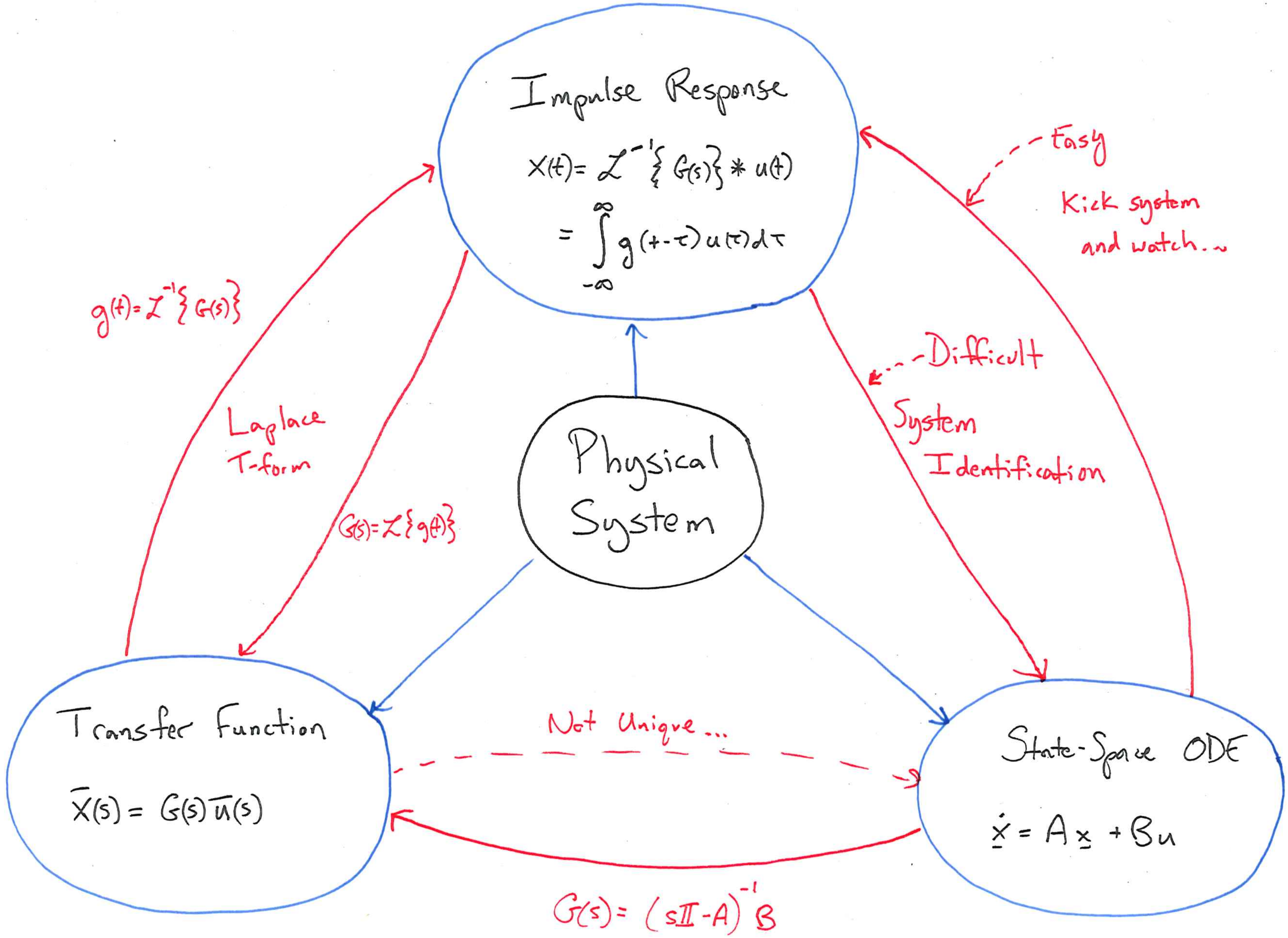
Time Domain:
Impulse Response

Finally, we may inverse Laplace transfor to find the solution in the time domain in terms of the impulse response $g(t) = \mathcal{L}^{-1}\{G(s)\}$:

$$x(t) = g(t) * u(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau$$

The time shifted impulse response $g(t,\tau) \triangleq g(t-\tau)$ is often called Green's Function.

This idea generalizes to PDEs quite nicely...



Equivalent Representations for Linear Systems