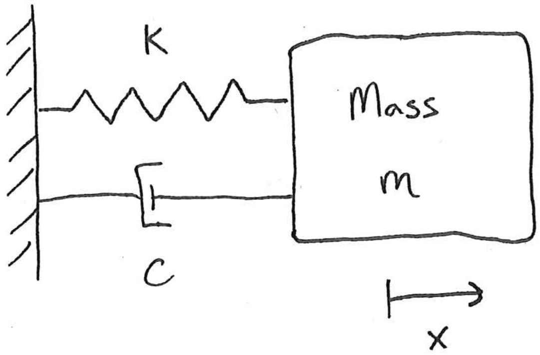
Overview of Topics

1. Laplace Transforms for ODEs with inputs & outputs
2. Transfer functions and frequency response
3. More on convolution

Revisit spring-mass-damper:



$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \underbrace{u(t)}_{\text{External Forcing}}$$

With numbers: $\ddot{x} + 5\dot{x} + 4x = u(t)$

$$\left.\begin{aligned} x(0) &= 2 \\ \dot{x}(0) &= -5 \end{aligned}\right] \text{ICs}$$

$$\mathcal{L}\{\cdot\} \Rightarrow \quad (s^2+5s+4)\,\bar{x}(s) = \overbrace{2s+5}^{\text{ICs}} + \underline{\bar{u}(s)}$$

$$\bar{x}(s) = \underline{\frac{2s+5}{s^2+5s+4}} + \underline{\frac{1}{s^2+5s+4}\,\bar{u}(s)}$$

$$\bar{X}(s) = \underline{\frac{1}{s+1} + \frac{1}{s+4}} + \underline{\frac{1}{s^2+5s+4}\,\bar{u}(s)}$$

convolution in time domain

$$x(t) = \underbrace{e^{-t} + e^{-4t}}_{\text{ICs}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+4}\right\} * u(t)}_{\text{Forcing}}$$

If zero initial conditions $(x(0)=0,\ \dot{x}(0)=0)$, then

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+4}\right\} * u(t)$$

Matlab: >> impulse(sys) will give the time history of $\mathcal{L}^{-1}\{sys\}$

If zero ICs:

$$\bar{x}(s) = \frac{1}{s^2+5s+4}\bar{u}(s) \Longrightarrow \frac{\bar{x}(s)}{\bar{u}(s)} = \underbrace{\frac{1}{s^2+5s+4}}_{\text{"Transfer function"}} \triangleq G(s)$$

How do we compute $\mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+4}\right\}$?

Partial Fractions: $\frac{1}{s^2+5s+4} = \frac{a}{s+1} + \frac{b}{s+4} = \frac{(a+b)s + 4a+b}{s^2+5s+4}$

$$\Longrightarrow \begin{aligned} &a=-b \\ &3a=1 \Longrightarrow a=1/3,\ b=-1/3 \end{aligned}$$

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}.$$

So $x(t) = \mathcal{L}^{-1}\{G(s)\} * u(t) = \frac{1}{3}\int_{-\infty}^{\infty}\left(e^{-\tau} - e^{-4\tau}\right)u(t-\tau)\,d\tau$

bounds on integration depend on when simulation or experiment started (i.e., when did forcing start?)

Matlab: >> x = lsim(sys, t, u);

First: >> s = tf('s')
>> sys = 1/(s^2+5s+4);

Lets look more at transfer function $G(s)$:

For <u>linear</u> systems, if we input a <u>sinusoid</u>, the output is also <u>sinusoidal</u>, but with a new <u>magnitude</u> and a new <u>phase</u>. (same frequency!)

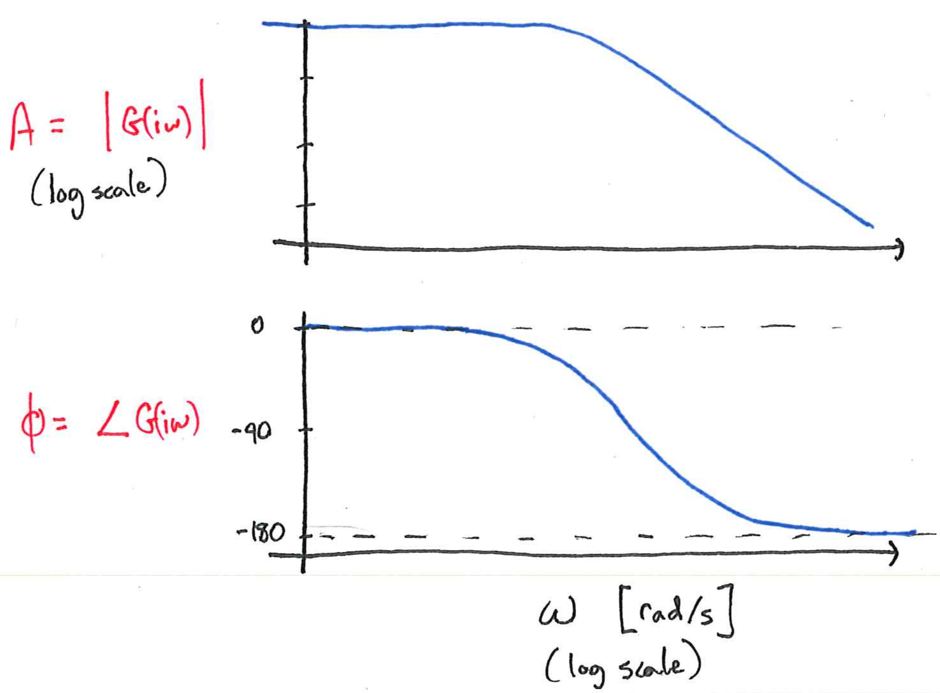
$$u(t) = \sin(\omega t) \longrightarrow \boxed{\underset{\text{transfer function}}{G(s)}} \longrightarrow \underset{\text{magnitude}}{A} \sin(\omega t + \underset{\text{phase}}{\phi})$$

system

The <u>magnitude</u> and <u>phase</u> are given by the transfer function evaluated at $s = i\omega$:

$G(i\omega)$ is a complex number.

$$\left.\begin{array}{l} A = |G(i\omega)| \text{ is the } \underline{\text{magnitude}} \\ \phi = \angle G(i\omega) \text{ is the } \underline{\text{phase}} \end{array}\right\} \text{These form a compelling plot...}$$



"Frequency Response"

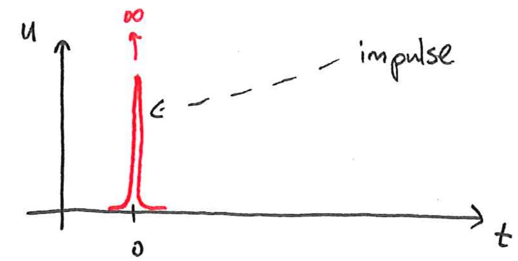
-OR-

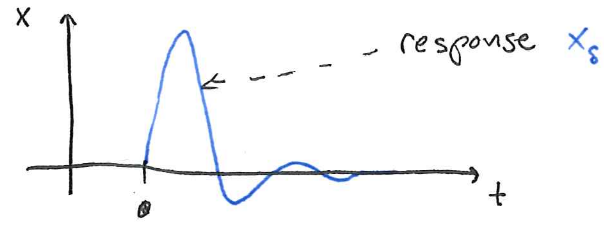
"Bode Plot"

```
>> bode(sys);
```

Impulse Response ...

Say $u(t) = \delta(t)$





For a given ODE, we will have a transfer function $G(s)$.

$\bar{u}(s) = 1$ for $u(t) = \delta$

$$\left.\begin{aligned} \bar{X}(s) &= G(s)\,\bar{u}(s) \\ \bar{X}_\delta(s) &= G(s) \cdot 1 \end{aligned}\right\} \Longrightarrow \quad x_\delta(t) = \mathcal{L}^{-1}\{G(s)\}$$

Impulse response $x_\delta(t)$ is the inverse Laplace transform of the transfer function!!

Or, we could work exclusively in time domain...

Example: $\dot{x} - \lambda x = u(t) = \delta(t)$ (assume zero ICs)

Integrate both sides: $\displaystyle\int_{-\varepsilon}^{\varepsilon} \dot{x}(t)\,dt - \lambda \int_{-\varepsilon}^{\varepsilon} x\,dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt$

$$x(\varepsilon) - x(-\varepsilon) = 1$$

If $x(0) = 0$, then $x(0^+) = 1$ ...

Impulse response is like starting the system off with an initial condition.

Matlab: `>> impulse (sys)`

More generally, consider a system of ODEs with input:

$$\overbrace{\dot{x} = A x}^{\text{dynamics}} + Bu \quad \text{input(s) forcing}$$

$$y = C x$$

output measurement(s)

internal system state

$$\mathcal{L}\{\cdot\} \Rightarrow \quad s\bar{x}(s) - x(0) = A\bar{x}(s) + B\bar{u}(s)$$

$$\bar{y}(s) = C\bar{x}(s)$$

$$(s\mathbb{I}-A)\,\bar{x}(s) = x(0) + B\bar{u}(s)$$

if $x(0)=0$ ... for simplicity

$$\bar{x}(s) = (s\mathbb{I}-A)^{-1}B\bar{u}(s)$$

$$\bar{y}(s) = C\bar{x}(s) = C(s\mathbb{I}-A)^{-1}B\bar{u}(s)$$

$$\Rightarrow \boxed{\frac{\bar{y}(s)}{\bar{u}(s)} = C(s\mathbb{I}-A)^{-1}B}$$

Input-Output Transfer function $G(s)$.

Spring-mass-damper

$$\begin{bmatrix} \dot{x} = v \\ \dot{v} = -\omega_0^2 x - \omega_0 \xi v + u(t) \end{bmatrix}$$

Measure position 'x'

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\xi\omega_0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{0}_{D} \cdot u(t)$$

Matlab: >> sys = ss(A,B,C,0);