Overview of Topics

1. Laplace Transforms and ODEs

$$\text{ODE} \xRightarrow{\mathcal{L}} \text{Algebraic Eq}^{\underline{n}}$$

Recall that $\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0)$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s^2\bar{f}(s) - sf(0) - f'(0)$$

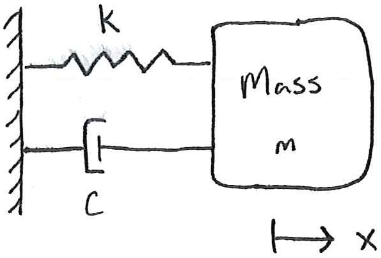
Each '0' is technically 0^-... the point just before $t=0$.

n^{th}-derivative

In general: $\mathcal{L}\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$.

Extremely useful for solving ODEs

Example: Spring-Mass-Damper



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\Downarrow$$

$$\boxed{\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0}$$

$\omega_0 = \sqrt{k/m}$

$\zeta = c/(2\sqrt{km})$

Lets solve for the following values:

$$\ddot{x} + 5\dot{x} + 4x = 0 \quad ; \quad x(0) = 2 \quad ; \quad \dot{x}(0) = -5$$

Laplace Transform:

$$\underbrace{s^2\bar{x}(s) - s\,x(0) - \dot{x}(0)}_{\mathcal{L}\{\ddot{x}\}} + \underbrace{5\left(s\bar{x}(s) - x(0)\right)}_{5\mathcal{L}\{\dot{x}\}} + \underbrace{4\,\bar{x}(s)}_{4\mathcal{L}\{x\}} = 0$$

Collect $\bar{x}(s)$ terms:

Initial Conditions (just numbers...)

$$\underbrace{\left(s^2 + 5s + 4\right)}_{\text{Polynomial in 's'}}\bar{x}(s) = \underbrace{s\left(x(0)\right) + 5x(0) + \dot{x}(0)}_{\text{Polynomial in 's'}}$$

(characteristic polynomial)
(dynamics)

Partial Fractions...

$$\bar{X}(s) = \frac{2s + 5}{s^2 + 5s + 4} = \frac{2s + 5}{(s+1)(s+4)} = \frac{1}{s+1} + \frac{1}{s+4}$$

Because linear superposition: $x(t) = \mathcal{L}^{-1}\{\bar{x}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$

$$\boxed{x(t) = e^{-t} + e^{-4t}}$$

Simple Example (Boyce & DiPrima)

$$\ddot{x} - \dot{x} - 2x = 0 \quad , \quad x(0) = 1 \quad , \quad \dot{x}(0) = 0$$

$$s^2 \bar{x}(s) - s\,x(0) - \dot{x}(0) - \left(s\bar{x}(s) - x(0) \right) - 2\bar{x}(s) = 0$$

$$\left(s^2 - s - 2\right) \bar{x}(s) = s - 1$$

$$\bar{x}(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}$$

$$= \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

so $a(s+1) + b(s-2) = s - 1$ (numerators must be equal...)

$$\left.\begin{array}{l} (a+b)s = s \\ a - 2b = -1 \end{array}\right\} \quad \left.\begin{array}{l} a + b = 1 \\ a - 2b = -1 \end{array}\right\} \quad \begin{array}{l} 3b = 2 \Rightarrow b = 2/3 \\ \Rightarrow a = 1/3 \end{array}$$

$$\bar{x}(s) = \frac{1}{3} \cdot \frac{1}{s-2} + \frac{2}{3} \frac{1}{s+1}$$

$$x(t) = \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3} \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{x(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}}$$

General Procedure:

I. Well-posed ODE with Initial Conditions

$$\begin{cases} x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0 \\ x(0) = b_0 \\ \dot{x}(0) = b_1 \\ \vdots \\ x^{(n-1)}(0) = b_{n-1} \end{cases}$$

II. Laplace tform all terms

$$\underbrace{\left[s^n\bar{x}(s) - \dots - b_{n-1}\right]}_{\mathcal{L}\{x^{(n)}\}} + a_{n-1}\underbrace{\left[s^{n-1}\bar{x}(s) - \dots - b_{n-2}\right]}_{\mathcal{L}\{x^{(n-1)}\}} + \dots + a_1\underbrace{\left[s\bar{x}(s) - b_0\right]}_{\mathcal{L}\{\dot{x}\}} + a_0\underbrace{\bar{x}(s)}_{\mathcal{L}\{x\}} = 0$$

III. Collect all terms with $\bar{x}(s)$ and separate from I.C. terms

$$\underbrace{\left(s^n + a_{n-1}s^{(n-1)} + \dots + a_2 s^2 + a_1 s + a_0\right)}_{\text{Characteristic polynomial captures all dynamics}} \bar{x}(s) = \underbrace{\text{polynomial in 's' with coefficients given by a's and b's}\dots}_{\text{This polynomial accounts for all initial conditions.}}$$

$$\triangleq P(s; a, b)$$

IV. Isolate solution $\bar{x}$

$$\bar{x}(s) = \frac{P(s; a,b)}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

find roots of denominator (i.e. eigenvalues)

solve for $c_1, c_2, \dots, c_n$ using method of partial fractions

V. Partial Fractions.

$$\bar{x}(s) = \frac{c_1}{(s+\lambda_1)} + \frac{c_2}{(s+\lambda_2)} + \dots + \frac{c_n}{(s+\lambda_n)}$$

VI. Solution

$$\boxed{x(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} + \dots + c_n e^{-\lambda_n t}}$$

Solution.

What if our ODE is forced?

$$\ddot{x} + 5\dot{x} + 4x = u(t)$$

$x(0) = 2$

$\dot{x}(0) = -5$

'forcing term' ... inhomogeneous ... basis for all of control theory!

$$\mathcal{L}\{\cdot\} \Rightarrow (s^2+5s+4)\bar{x}(s) = \underbrace{2s+5}_{\text{Initial Conditions}} + \underbrace{\bar{u}(s)}_{\text{Forcing!}}$$

$$\bar{X}(s) = \frac{2s+5}{s^2+5s+4} + \frac{\bar{u}(s)}{s^2+5s+4}$$

$$G(s) = \frac{1}{s^2+5s+4}$$ is called the "transfer function"

$$\bar{x}(s) = \frac{1}{s+1} + \frac{1}{s+4} + G(s)\,\bar{u}(s)$$

$$x(t) = \underbrace{e^{-t} - e^{-4t}}_{\text{Homogeneous sol}^{n}} + \underbrace{\mathcal{L}^{-1}\{G(s)\} * u(t)}_{\text{Particular sol}^{n}\ldots}$$