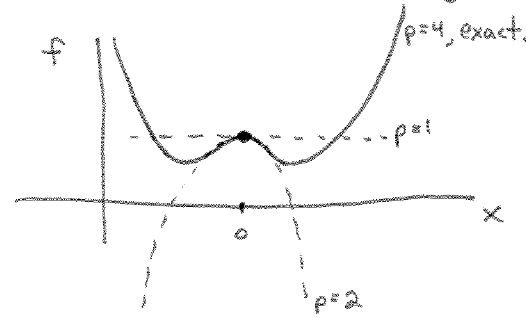
Discrete Fourier Transform

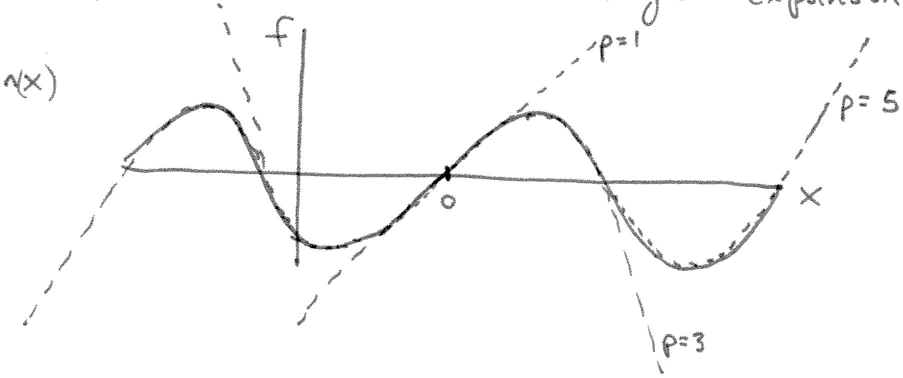
- Some functions are well represented by a Talor expansion:

$f(x) = x^4 - x^2$



- some functions are ill-suited for Taylor expansion:

$f(x) = \sin(x)$



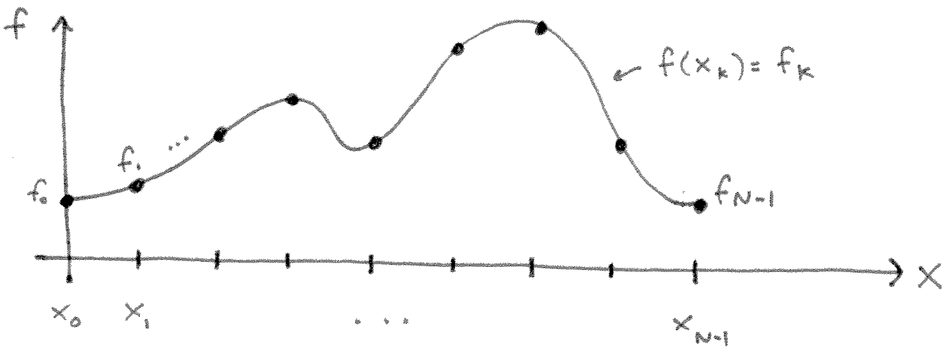
However, instead of expanding a function in a polynomial basis (as in Taylor expansion)

it is possible to expand in a basis of sin or cos functions of various frequencies (Fourier series).

- only need a single sin to describe the second function above.

There is an entire theory of expanding functions in terms of $\sin(k\omega x)$ called Fourier Analysis.

We are interested in extracting frequencies from <u>DATA</u>:



$$\{f_0, f_1, \ldots, f_{N-1}\} \xrightarrow{\text{DFT}} \{\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{N-1}\}.$$

DFT: $$\hat{f}_k = \sum_{n=0}^{N-1} f_n \cdot e^{-i2\pi kn/N}.$$ (discrete fourier coefficients)

$\hat{f}$'s can be used to re-construct f's by:

iDFT: $$f_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}_k \cdot e^{i2\pi kn/N}$$

Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Let $\omega_N = e^{-2\pi i/N}$

Then we may compute DFT by matrix multiplication:

$$\underbrace{\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}}_{\text{Fourier Coefficients OUT}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{bmatrix}}_{\underline{\underline{F}} \text{ is a Vandermonde matrix}} \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix}}_{\text{Data IN}}$$