Overview of Topics

1. Properties of Fourier Transforms
2. Examples

Properties of Fourier Transform :

A. $\mathcal{F}$ of derivative of function :

(integration by parts)

$$\mathcal{F}\left(\frac{d}{dx} f(x)\right) = \int_{-\infty}^{\infty} \underbrace{f'(x)}_{dV} \underbrace{e^{-iwx}}_{u} \, dx$$

$$= \underbrace{\underbrace{\left[f(x)\, e^{-iwx}\right]_{-\infty}^{\infty}}_{uv}} - \int_{-\infty}^{\infty} \underbrace{f(x)}_{v} \underbrace{\left[-iw\, e^{-iwx}\right]}_{du} dx$$

$f \to 0$ for $x \to \pm\infty$
or else $\mathcal{F}$ doesn't converge
... $|e^{iwx}| = 1$ even as $x \to \pm\infty$.

$$= iw \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \, dx$$

$$\boxed{\mathcal{F}\left(\frac{d}{dx} f(x)\right) = iw\, \mathcal{F}(f(x))}$$

Very important... we will use this to turn PDEs into ODEs
very much like separation of variables...

$$u_{tt} = c\, u_{xx} \xrightarrow{\mathcal{F}} \hat{u}_{tt} = -c w^2 \hat{u}$$

PDE! ODE!

Properties of Fourier Transform:

B. $\mathcal{F}$ and convolution $*$:

Define convolution of two functions $(f*g)$:

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$$

$f*g = g*f$ (check this!)

Let $\hat{f} = \mathcal{F}(f(x))$ and $\hat{g} = \mathcal{F}(g(x))$

$$\mathcal{F}^{-1}(\hat{f}(\omega)\hat{g}(\omega))(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,\hat{g}(\omega)\,e^{i\omega x}\,d\omega = \int_{-\infty}^{\infty} \hat{f}e^{i\omega x}\left(\frac{1}{2\pi}\int_{-\infty} g(y)e^{-i\omega y}\,dy\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)\,\hat{f}(\omega)\,e^{i\omega(x-y)}\,d\omega\,dy$$

$$= \int_{-\infty}^{\infty} g(y)\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega(x-y)}\,d\omega\right)}_{f(x-y)}dy$$

$$= \int_{-\infty}^{\infty} g(y)\,f(x-y)\,dy = g*f = f*g$$

$$\boxed{\begin{aligned}\mathcal{F}(f*g) &= \hat{f}\hat{g}\\ \mathcal{F}^{-1}(\hat{f}\hat{g}) &= f*g\end{aligned}}$$

"multiplying functions in the frequency domain is same as convolving in the spatial domain"

Other Properties of Fourier Transform:

C. Linearity:

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$$

$$\mathcal{F}^{-1}(\alpha \hat{f} + \beta \hat{g}) = \alpha \mathcal{F}^{-1}(\hat{f}) + \beta \mathcal{F}^{-1}(\hat{g}).$$

D. Shift in x and w:

$$f(x-a) = \mathcal{F}^{-1}\left(e^{-iaw}\hat{f}(w)\right)$$

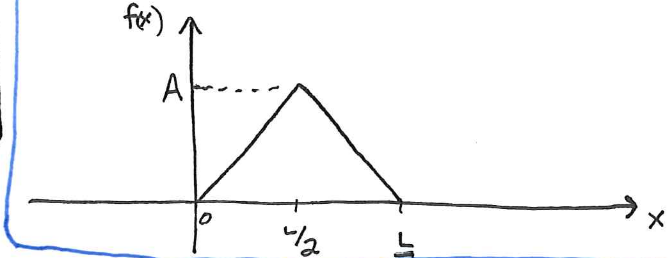
$$\mathcal{F}^{-1}\left(\hat{f}(w-b)\right) = e^{ibx} f(x)$$

E. Parseval's Theorem ("Z"'s theorem~?)

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2 \, dw = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

In other words, Fourier Transform preserves the L^2 norm (up to 2π constant).

Example: $\mathcal{F}(f(x))$ where $f(x) = \begin{cases} 0, & x<0 \\ \dfrac{2Ax}{L}, & x \in [0, L/2) \\ \dfrac{2A}{L}(L-x), & x \in [L/2, L] \\ 0, & x > L \end{cases}$



$$\mathcal{F}(f(x)) = \int_0^{L/2} \frac{2A}{L} x e^{-i\omega x}\,dx + \int_{L/2}^{L} \frac{2A}{L}(L-x) e^{-i\omega x}\,dx$$

$$= \frac{2A}{L}\left[\int_0^{L/2} x e^{-i\omega x}\,dx + \int_{L/2}^{L} L e^{-i\omega x}\,dx + \int_{L/2}^{L} -x e^{-i\omega x}\,dx\right]$$

$$= \frac{2A}{L}\left[\left[\frac{(1+i\omega x)}{\omega^2} e^{-i\omega x}\right]_0^{L/2} - \left[\frac{(1+i\omega x)}{\omega^2} e^{-i\omega x}\right]_{L/2}^{L} + \left[\frac{L e^{-i\omega x}}{-i\omega}\right]_{L/2}^{L}\right]$$

$$= \frac{2A}{\omega^2 L}\left[e^{-i\omega L/2}(1+i\omega L/2) - 1 - e^{-i\omega L}(1+i\omega L) + e^{-i\omega L/2}(1+i\omega L/2) + i\omega L\left(e^{-i\omega L} - e^{-i\omega L/2}\right)\right]$$

$$= \frac{2A}{\omega^2 L}\left[2e^{-i\omega L/2} - 1 - e^{-i\omega L}\right]$$

$$= \frac{4A}{\omega^2 L}\left[1 - \underbrace{\frac{e^{i\omega L/2} + e^{-i\omega L/2}}{2}}_{\cos(\omega L/2)}\right] e^{-i\omega L/2}$$

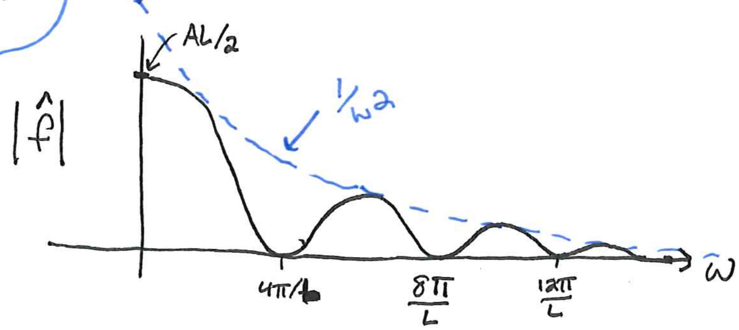
$$= \frac{4A}{\omega^2 L}\left[1 - \cos\left(\omega L/2\right)\right] e^{-i\omega L/2}$$

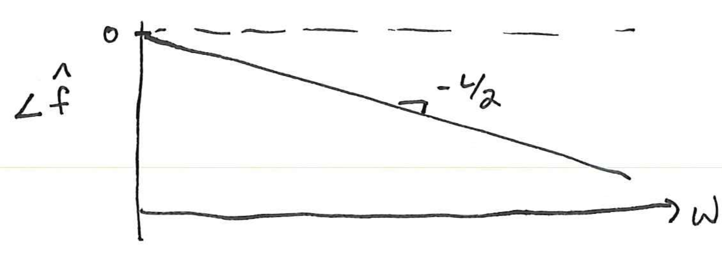
$$\boxed{\mathcal{F}(f) = \frac{8A}{\omega^2 L} \sin^2\left(\frac{\omega L}{4}\right) e^{-i\omega L/2}}$$

Let's plot magnitude and phase of $\hat{f}$:

$$|\hat{f}| = \frac{8A}{\omega^2 L} \sin^2\left(\frac{\omega L}{4}\right)$$

$$\angle \hat{f} = -\omega L/2$$





Auto correlation $R_{ff}(\tau)$:

$$R_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, f(t+\tau)\, dt$$ Auto correlation

$$S_{ff}(\omega) = \mathcal{F}(R_{ff}) = \int_{-\infty}^{\infty} R_{ff}\, e^{-i\omega\tau}\, d\tau$$ Power Spectrum

Note: $R_{ff} = \mathcal{F}^{-1}(S_{ff})$ ...

Fourier Transform of (very special) functions, such as sin and cos that are not L^1 (i.e. they don't have $\int_{-\infty}^{\infty} \cdots$)

Idea: define a <u>windowed</u> F.T. and take limit as window $\to \infty$ large.

$$f(x) = \sin(\Omega x) \Longrightarrow \underset{\text{"L"}}{\hat{f}_L(\omega)} = \int_{-L/2}^{L/2} \sin(\Omega x)\, e^{-i\omega x}\, dx$$

$$= \int_{-L/2}^{L/2} \Big[\underbrace{\sin(\Omega x)\cos(\omega x)}_{\text{odd } = 0} - i \underbrace{\sin(\Omega x)\sin(\omega x)}_{\text{even}}\Big]\, dx$$

$$= -\frac{i}{2}\int_{-L/2}^{L/2} \Big[\cos\big((\Omega-\omega)x\big) - \cos\big((\Omega+\omega)x\big)\Big]\, dx$$

$$= -\frac{i}{2}\left[\frac{\sin\big((\Omega-\omega)x\big)}{\Omega-\omega} - \frac{\sin\big((\Omega+\omega)x\big)}{\Omega+\omega}\right]_{-L/2}^{L/2}$$

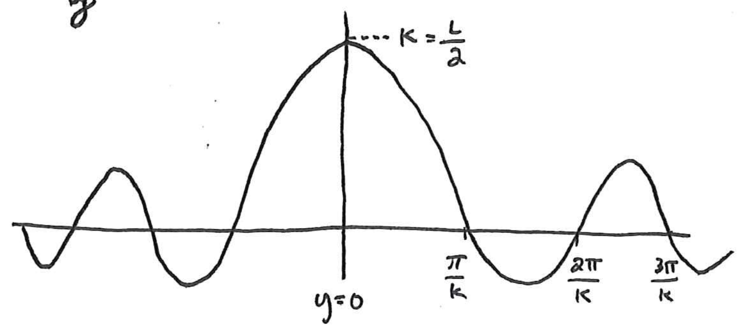
$$\boxed{\tilde{\mathcal{F}}_L\big(\sin(\Omega x)\big) = -i\left[\frac{\sin\big((\Omega-\omega)\frac{L}{2}\big)}{\Omega-\omega} - \frac{\sin\big((\Omega+\omega)\frac{L}{2}\big)}{\Omega+\omega}\right]}$$

See next page for plots

and $\lim_{L\to\infty} \ldots$

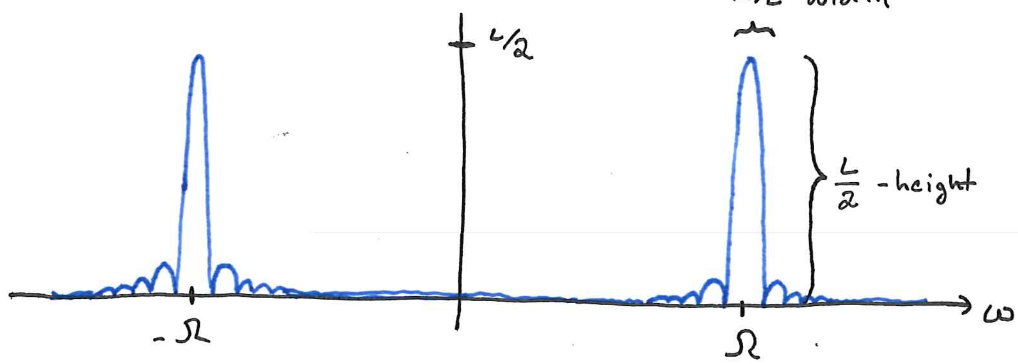
$$\mathcal{F}_L\left(\sin(\Omega x)\right) = -i\left[\frac{\sin\left((\Omega-\omega)\frac{L}{2}\right)}{\Omega-\omega} - \frac{\sin\left((\Omega+\omega)\frac{L}{2}\right)}{\Omega+\omega}\right]$$

$$\frac{\sin\left((\Omega-\omega)\frac{L}{2}\right)}{\Omega-\omega} = \frac{\sin(ky)}{y} \quad \left(\text{for } y = \Omega-\omega \text{ and } k = L/2\right)$$



width of main peak is $\frac{4\pi}{L}$

So $\left|\hat{f}_L(\omega)\right|$ is

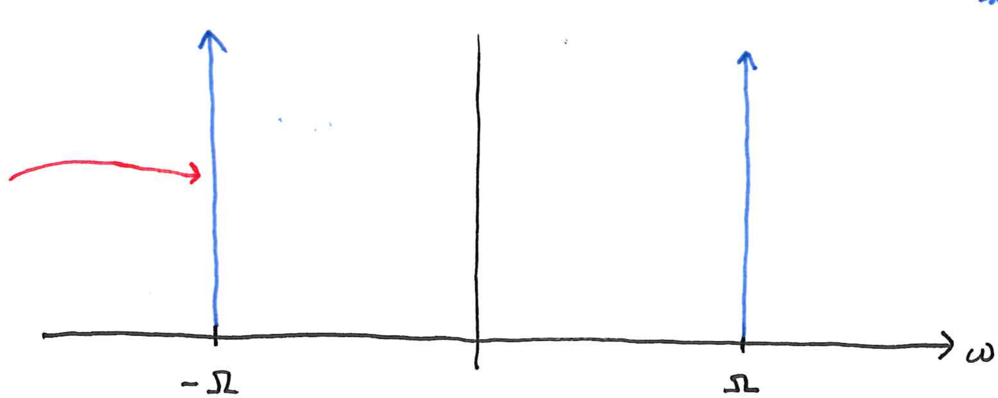


If we take $L \to \infty$

$$\lim_{L\to\infty} \hat{f}_L(\omega) = -i\left[\pi\,\delta(\Omega-\omega) - \pi\,\delta(\Omega+\omega)\right]$$

Note: factor of π because area of $\int_{-\infty}^{\infty} \frac{\sin ky}{y} = \pi$

Dirac δ function



$\int \delta(x)\,dx = 1$