L14: Feb. 6, 2015 ME 565, Winter 2015

Overview of Topics

(1) Fourier Transform Integral

- limit of Fourier Series on $[-L/2, L/2]$ as $L \to \infty$

(2) Properties of F.T.

So we have e^{inx} for $n \in \mathbb{Z}$ (for integer n)

as a basis for periodic, complex-valued functions

on interval $[0, 2\pi]$.

Also, $e^{2inx\pi/L}$ basis for $L^2([0, L])$.

Notation: $L^2([a,b])$

The space of all square integrable functions on $x \in [a,b]$.

Square integrable: $\int_a^b f(x)^2 \, dx = \underline{\underline{\text{finite}}}.$

Example of non-square-integrable function: $f(x) = 1/x$ on $[0, 2]$.

$\{\phi_n = e^{inx}\}_{n=-\infty}^{\infty}$ is orthogonal function basis...

$$\langle \phi_m, \phi_n \rangle = \int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \left[\frac{e^{i(m-n)x}}{i(m-n)} \right]_{-\pi}^{\pi}$$

inner product.

$$= \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}.$$

$$\langle \phi_m, \phi_n \rangle = 2\pi \, \delta_{mn}$$

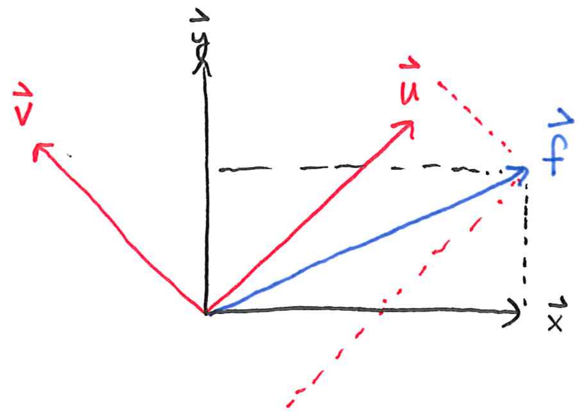
Kronecker Delta

conjugate

$\mathbb{C}$-valued ... ~~transpose~~

$$\langle \phi_m, \phi_n \rangle = \int_{-\pi}^{\pi} \phi_m(x) \, \overline{\phi_n(x)} \, dx$$

Write a vector in a new coordinate system:



$$\vec{f} = \langle \vec{f}, \vec{x} \rangle \vec{x} + \langle \vec{f}, \vec{y} \rangle \vec{y}$$

$$= \langle \vec{f}, \vec{u} \rangle \vec{u} + \langle \vec{f}, \vec{v} \rangle \vec{v}$$

A Fourier Series is just a change of coordinates of a function $f(x)$ into an infinite dimensional orthogonal function space spanned by sines & cosines:

i.e. $\phi_n = e^{inx} = \cos(nx) + i\sin(nx)$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \langle f(x), \phi_n(x) \rangle \, \phi_n(x)}$$

```
clear all, close all, clc

x = 0:.001:1;
for k=1:100
f = sin(pi*x);
g = cos(k*pi*x);
ksum(k) = sum(f.*g);
end
plot(ksum)
% sum(f.*g)
```

Fourier Series on $[-L, L]$ as $L \to \infty$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \qquad \left(\text{for } \mathbb{R}\text{-valued } f,\ c_{-n} = \overline{c_n}\right)$$

with $$c_n = \frac{1}{2L} \langle f(x), \phi_n \rangle = \frac{1}{2L} \int_{-L}^{L} f(x) \underbrace{e^{-in\pi x/L}}_{\overline{\phi_n(x)}\ \text{(complex conjugate)}} dx$$

$$\boxed{f(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \langle f(x), \phi_n(x) \rangle \phi_n(x)}$$ ☆ Change of Coordinates

f(x) is represented by a sum of sines and cosines (or alternatively by complex amplitudes c_n ... magnitude and phase) of discrete set of frequencies:

$$\boxed{\omega_n = \frac{n\pi}{L}}.$$

Now, take $L \to \infty$ and ω spacing $\to 0$...

Define $\omega_n = \frac{n\pi}{L}$, $\Delta\omega = \pi/L$, and take $\lim_{L\to\infty}$ $\left(\lim_{\Delta\omega\to 0}\right)$

$$f(x) = \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \underbrace{\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi)\, e^{-i\overbrace{n\Delta\omega}^{n\Delta\omega=\omega}\xi}\, d\xi}_{=\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi \,\triangleq\, \hat{f}(\omega) = \hat{f}(n\Delta\omega)}\, e^{in\Delta\omega x}$$

Definition of Fourier Transform!

$$f(x) = \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \hat{f}(\omega)\, e^{in\Delta\omega x}$$

$$= \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \hat{f}(\omega)\, e^{i\omega x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

$$\boxed{\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega \end{aligned}}$$

$\hat{f}(\omega) = \mathcal{F}(f(x))$

$f(x) = \mathcal{F}^{-1}(\hat{f}(\omega))$

Fourier Transform Pair!

Technical Note: These integrals converge as long as

$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

and $\int_{-\infty}^{\infty} |\hat{f}(\omega)|\,d\omega < \infty$

i.e. $f, \hat{f} \in L^1(-\infty, \infty)$ ...