L13 : Feb. 4, 2015

ME 565, Winter 2015

Overview of Topics

① Infinite Dimensional Function Spaces

(a) Vector Space

(b) Inner Product

(c) Norm (distance).

② Fourier Series as ∞-dimensional change of coordinates...

(a) basis functions $\phi_n = e^{inx}$ ...

Vector Space (in $\mathbb{R}$ or $\mathbb{C}$)

A set of "vectors" V, that satisfy eight axioms:

(there is also a notion of addition & scalar multiplication)

1. Associativity $\quad u+(v+w) = (u+v)+w$
2. Commutativity $\quad u+v = v+u$
3. Identity (addition) $\quad 0+v = v$ $\quad$ (there is such a '0' vector)
4. Additive inverse $\quad v+(-v) = 0$ $\quad$ (there is an 'inverse' $-v$ $\forall v$)
5. Consistent scalar multiplication $\quad a(bv) = (ab)v$
6. Identity (multiplication) $\quad 1v = v$
7. Distributive 1 $\quad a(u+v) = au + av$
8. Distributive 2 $\quad (a+b)v = av + bv$

$a, b \in \mathbb{R}$ or $\in \mathbb{C}$ $\quad$ (numbers)

$u, v, w \in V$ $\quad$ (vectors)

L^2 inner product space: (i.e. function space w/ ∞ dimensions)

An inner-product space is a vector space that has additional structure in the form of an inner product:

$\langle \cdot , \cdot \rangle$ maps two vectors to the reals (or Complex) numbers

Conjugate $\quad \langle f, g \rangle = \overline{\langle g, f \rangle}$

Linearity $\quad \langle \alpha f, g \rangle = \alpha \langle f, g \rangle \qquad (\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C})$

$\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

Positive Definite $\quad \langle f, f \rangle \geq 0 \qquad$ (with $=0$ only if $f=0$)

L^2 inner product on functions:

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, \overline{g(x)}\, dx$$

$L^2([a,b])$ is the space of all functions $f(x)$ that have finite L^2 inner product on $[a,b]$:

$$\langle f(x), f(x) \rangle = \int_a^b f(x)\, \overline{f(x)}\, dx = \underline{\text{finite}}.$$

Inner Products also imply a Norm, which allows us to measure distance:

$$\|f\|_2 = \left(\langle f, f\rangle_2\right)^{1/2}$$

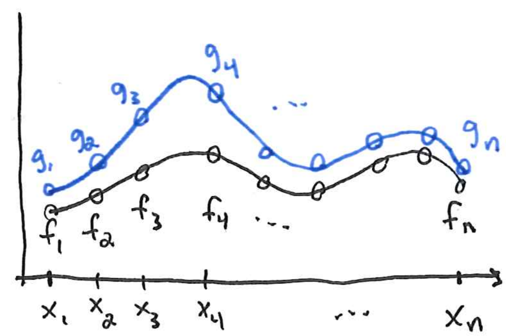
$$= \sqrt{\langle f, f\rangle}$$

$$= \left(\int_a^b f(x)\,\overline{f(x)}\,dx\right)^{1/2}$$

Distance between two real-valued functions $f(x)$ and $g(x)$:

$$\|f - g\|_2 = \left(\int_a^b (f(x) - g(x))^2\,dx\right)^{1/2}$$

Completely consistent with vector norm if $\vec{f}$ and $\vec{g}$ were discretized vectors:



$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix}$$

$$\|\vec{f} - \vec{g}\|_2 = \left(\sum_{k=1}^{n} (f_k - g_k)^2\right)^{1/2}$$

$$= \sqrt{(f_1 - g_1)^2 + (f_2 - g_2)^2 + \dots + (f_n - g_n)^2}$$

Because we have 'cos' and 'sin', we may try to write Fourier Series in Complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \text{where} \qquad c_n = \alpha_n + i\beta_n$$

$$= \sum_{n=-\infty}^{\infty} (\alpha_n + i\beta_n)(\cos(nx) + i\sin(nx))$$

$$= (\alpha_0 + i\beta_0) + \sum_{n=1}^{\infty}\Big[(\alpha_{-n} + i\beta_{-n})(\cos(nx) - i\sin(nx)) + (\alpha_n + i\beta_n)(\cos(nx) + i\sin(nx))\Big]$$

$$= (\alpha_0 + i\beta_0) + \sum_{n=1}^{\infty}\Big[(\alpha_{-n} + \alpha_n)\cos(nx) + (\beta_{-n} - \beta_n)\sin(nx)\Big]$$

$$+ i\sum_{n=1}^{\infty}\Big[(\beta_{-n} + \beta_n)\cos(nx) - (\alpha_{-n} - \alpha_n)\sin(nx)\Big]$$

If $f(x)$ is real-valued, then

$$\beta_{-n} = -\beta_n$$
$$\alpha_{-n} = \alpha_n$$

So $c_{-n} = \overline{c_n}$ (complex conjugate)

So we have e^{inx} for $n \in \mathbb{Z}$ (for integer n)

as a basis for periodic, complex-valued functions

on interval $[0, 2\pi]$.

Also, $e^{2inx\pi/L}$ basis for $L^2([0,L])$.

Notation: $L^2([a,b])$

The space of all square integrable functions on $x \in [a,b]$.

Square integrable : $\int_a^b f(x)^2\,dx = \underline{\text{finite}}$.

Example of non-square-integrable function : $f(x) = 1/x$ on $[0, 2]$.

$\{\phi_n = e^{inx}\}_{n=-\infty}^{\infty}$ is orthogonal function basis...

$$\langle \phi_m, \phi_n \rangle = \int_{-\pi}^{\pi} e^{imx} e^{-inx}\,dx = \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \left[\frac{e^{i(m-n)x}}{i(m-n)}\right]_{-\pi}^{\pi}$$

inner product.

$$= \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}.$$

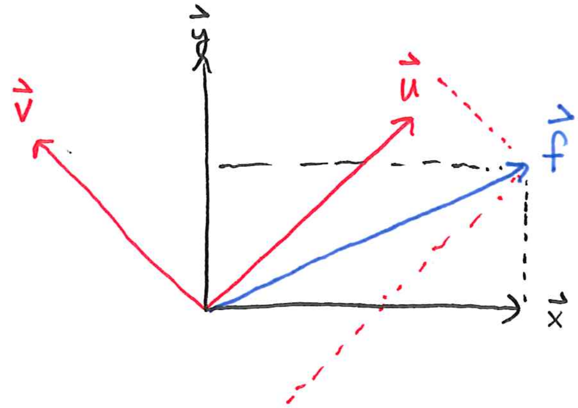
$$\langle \phi_m, \phi_n \rangle = 2\pi\,\delta_{mn}$$

Kronecker Delta

$\mathbb{C}$-valued ... ~~transpose~~ conjugate

$$\langle \phi_m, \phi_n \rangle = \int_{-\pi}^{\pi} \phi_m(x)\,\overline{\phi_n(x)}\,dx$$

Write a vector in a new coordinate system:



$$\vec{f} = \langle \vec{f}, \vec{x} \rangle \vec{x} + \langle \vec{f}, \vec{y} \rangle \vec{y}$$

$$= \langle \vec{f}, \vec{u} \rangle \vec{u} + \langle \vec{f}, \vec{v} \rangle \vec{v}$$

A Fourier Series is just a change of coordinates of a function $f(x)$ into an infinite dimensional orthogonal function space spanned by sines & cosines:

i.e. $\phi_n = e^{inx} = \cos(nx) + i\sin(nx)$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \langle f(x), \phi_n(x) \rangle \, \phi_n(x)}$$

```
clear all, close all, clc

x = 0:.001:1;
for k=1:100
f = sin(pi*x);
g = cos(k*pi*x);
ksum(k) = sum(f.*g);
end
plot(ksum)
% sum(f.*g)
```