Overview of Topics

1. Fourier ~~Transform~~ Series
   (a) Definition
   (b) Properties
   (c) Examples

2. Orthogonal Function Spaces

Fourier Series : $x \in [-\pi, \pi]$

If $f(x)$ is 2π periodic and piecewise smooth, then

$$f(x) = f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

There are some caveats associated with this equal sign:

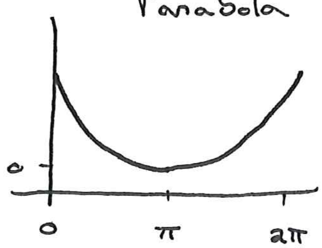
1. True in L_2-integral sense:

$$\|f - f_{FS}\|_2 = \int_0^{2\pi} (f - f_{FS})^2\, dx = 0$$

2. At discontinuities, $f_{FS}(x)$ cannot exactly $= f(x)$.

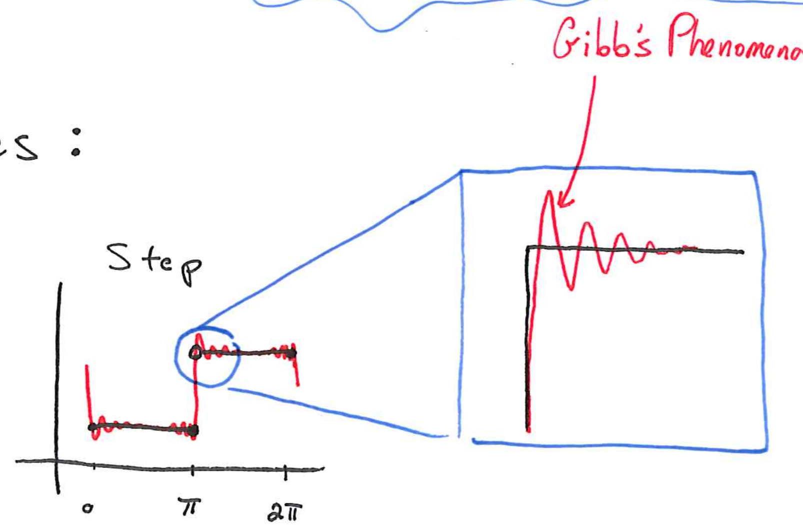
Try on some shapes:

Parabola



$f(x) = (x - \pi)^2$ on $x \in [0, 2\pi)$

Step



$$f(x) = \begin{cases} 0 & , 0 \le x \le \pi \\ 1 & , \pi < x < 2\pi \end{cases}$$

Note, not 2π periodic!!

Fourier Series for L-periodic functions

are easy too: $x \in [0, L)$

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n x}{L}\right) dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi n x}{L}\right) dx$$

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clear all, close all, clc

dx = 0.005;
L = 1;
x = dx:dx:L;

f = ones(size(x));
f(1:length(f)/2) = 0*f(1:length(f)/2);
%
f = sin(x);
%
% f = (x-pi).^2;
%
% f1 = (1:1:25)/25;
% f2 = (25:-1:1)/25;
% f3 = -1*f1;
% f4 = -1*f2;
% f = [f1 f2 f3 f4 f1 f2 f3 f4];

fFS = zeros(size(x));

A0 = (2/L)*sum(f.*ones(size(x)))*dx;
for m=1:100
    fFS = A0/2;


    for k=1:m
        Ak = (2/L)*sum(f.*cos(2*pi*k*x/L))*dx;
        Bk = (2/L)*sum(f.*sin(2*pi*k*x/L))*dx;
        fFS = fFS + Ak*cos(2*k*pi*x/L) + Bk*sin(2*k*pi*x/L);
    end

    plot(x,f,'k')
    hold on
    plot(x,fFS,'r--')

    pause(0.1)
end
```

Because we have 'cos' and 'sin', we may try to write Fourier Series in Complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \text{where} \quad c_n = \alpha_n + i\beta_n$$

$$= \sum_{n=-\infty}^{\infty} (\alpha_n + i\beta_n)(\cos(nx) + i\sin(nx))$$

$$= (\alpha_0 + i\beta_0) + \sum_{n=1}^{\infty} \Big[(\alpha_{-n} + i\beta_{-n})(\cos(nx) - i\sin(nx)) + (\alpha_n + i\beta_n)(\cos(nx) + i\sin(nx))\Big]$$

$$= (\alpha_0 + i\beta_0) + \sum_{n=1}^{\infty} \Big[(\alpha_{-n} + \alpha_n)\cos(nx) + (\beta_{-n} - \beta_n)\sin(nx)\Big]$$

$$+ i \sum_{n=1}^{\infty} \Big[(\beta_{-n} + \beta_n)\cos(nx) - (\alpha_{-n} - \alpha_n)\sin(nx)\Big]$$

If $f(x)$ is real-valued, then
$$\beta_{-n} = -\beta_n$$
$$\alpha_{-n} = \alpha_n$$

So $c_{-n} = \overline{c_n}$ (complex conjugate)

So we have e^{inx} for $n \in \mathbb{Z}$ (for integer n)

as a basis for periodic, complex-valued functions

on interval $[0, 2\pi]$.

Also, $e^{2inx\pi/L}$ basis for $L^2([0,L])$.

Notation: $L^2([a,b])$

The space of all square integrable functions on $x \in [a,b]$.

Square integrable: $\int_a^b f^2(x)\,dx = \underline{\text{finite}}$.

Example of non-square-integrable function: $f(x) = 1/x$ on $[0,2]$.

$\{\phi_n = e^{inx}\}_{n=-\infty}^{\infty}$ is orthogonal function basis...

$$\langle \phi_m, \phi_n \rangle = \int_{-\pi}^{\pi} e^{imx} e^{-inx}\,dx = \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \left[\frac{e^{i(m-n)x}}{i(m-n)}\right]_{-\pi}^{\pi}$$

inner product.

$$= \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n. \end{cases}$$

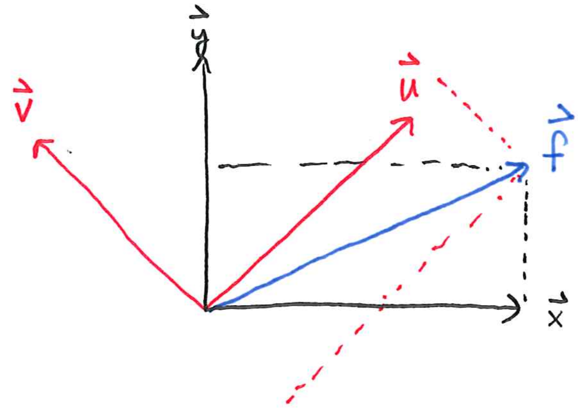
$\mathbb{C}$-valued ... ~~transpose~~ conjugate

$$\langle \phi_m, \phi_n \rangle = \int_{-\pi}^{\pi} \phi_m(x)\,\overline{\phi_n(x)}\,dx$$

$$\langle \phi_m, \phi_n \rangle = 2\pi\,\delta_{mn}$$

Kronecker Delta

Write a vector in a new coordinate system:



$$\vec{f} = \langle \vec{f}, \vec{x} \rangle \vec{x} + \langle \vec{f}, \vec{y} \rangle \vec{y}$$

$$= \langle \vec{f}, \vec{u} \rangle \vec{u} + \langle \vec{f}, \vec{v} \rangle \vec{v}$$

A Fourier Series is just a change of coordinates of a function $f(x)$ into an infinite dimensional orthogonal function space spanned by sines & cosines:

i.e. $\phi_n = e^{inx} = \cos(nx) + i\sin(nx)$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \langle f(x), \phi_n(x) \rangle \, \phi_n(x)}$$

```
clear all, close all, clc

x = 0:.001:1;
for k=1:100
f = sin(pi*x);
g = cos(k*pi*x);
ksum(k) = sum(f.*g);
end
plot(ksum)
% sum(f.*g)
```