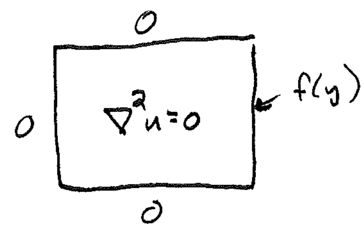
Overview of Topics

① Solve Laplace's Equation in 2D on rectangle



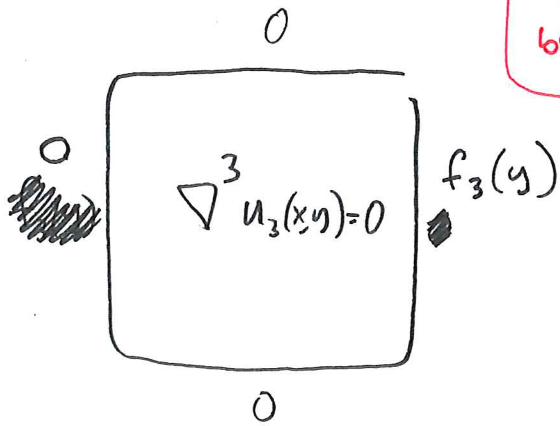
a. Separation of Variables

b. Boundary Conditions

c. Full Solution

Note: I decided to solve [diagram] f instead of f [diagram] because it is cleaner...

$$\nabla^2 u_3 = 0$$



$$\left.\begin{array}{ll} \text{BC1} & u_3(x,0) = 0 \\ \text{BC2} & u_3(x,H) = 0 \\ \text{BC3} & u_3(0,y) = 0 \end{array}\right\} \text{homogeneous BCs}$$

BC4 $\quad u_3(L,y) = f_3(y)$

Mark Twain says: "Damn important!"

We will solve this using a very important method called SEPARATION OF VARIABLES:

① Assume $u_3(x,y) = F(x)\,G(y)$ (big assumption?!)

② Try to use Eq^n ($\nabla^2 u_3 = 0$) and BC's (1-4) to determine $F(x)$ and $G(y)$.

$$\nabla^2 u_3 = \frac{\partial^2}{\partial x^2}\Big[F(x)\,G(y)\Big] + \frac{\partial^2}{\partial y^2}\Big[F(x)\,G(y)\Big] = 0$$

$$\Longrightarrow \boxed{G(y)\,F_{xx}(x) + F(x)\,G_{yy}(y) = 0}$$

$$\Longrightarrow \boxed{\frac{F_{xx}}{F(x)} = -\frac{G_{yy}}{G(y)}} = \text{constant!}$$

(why?)

Separation of Variables:

$$u(x,y) = F(x)\,G(y)$$

$$\nabla^2 u = G F_{xx} + F G_{yy} = 0$$

$$\boxed{\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = \text{constant}}$$

Jacob Bernoulli invented this method to solve the isochrone problem in 1690.

Why do these fractions equal a constant:

F and F_{xx} are both functions only of x

G and G_{yy} are both functions only of y.

So if $\text{func}(x) = \text{func}(y)$,

then they must both equal a constant!

Any constant will work to find a sol$^{\underline{n}}$ to $\nabla^2 u = 0$.

However, to satisfy boundary conditions, only certain constants will give $F(x)$ and $G(y)$ that satisfy the BCs...

$$u_3(x,y) = F(x)\, G(y)$$

From BC 1-3 :

$$u_3(x,0) = 0 \quad \Rightarrow \quad G(0) = 0$$

$$u_3(x,H) = 0 \quad \Rightarrow \quad G(H) = 0$$

$$u_3(0,y) = 0 \quad \Rightarrow \quad F(0) = 0$$

The function $G(y)$ might be easier to solve, since it has homogeneous BCs ...

Now, we use $\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = K$ to find functions F and G that satisfy boundary conditions!

Sep. of. Vars. results in two ODEs!

We know how to Solve & interpret these !!

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = \lambda$$

BCs only satisfied for some λ!

$$F_{xx} = \lambda F \quad ; \quad F(0) = 0, \quad F(L) = f_3(y)$$

ODE 1 for $F(x)$

$$G_{yy} = -\lambda G \quad ; \quad G(0) = G(H) = 0$$

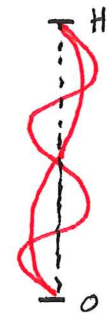
ODE 2 for $G(y)$

Big Picture:

① Solve ODE2 first for $G(y)$ since these BCs are easy.

Functions $G(y)$ will be sines with nodes at $y=0$ and $y=H$

i.e. $\lambda = \left(\frac{n\pi}{H}\right)^2$ for $n = 1, 2, 3, \ldots$



② Use these λ's in ODE1 to find "F" eigenfunctions

Functions will look like $e^{\lambda x} + e^{-\lambda x}$ ... i.e. $\sinh(x\lambda)$

The coefficients of these sinh's are chosen to satisfy right BC $F(x) = f_3(y)$

$\Downarrow$

This will lead to a Fourier Series ... soon.

$$G_{yy} = -\lambda G \; ; \quad G(0) = G(H) = 0$$

Case 1 (interesting): $\lambda > 0$

$G_{yy} = -\lambda G$ has sines & cosines for solutions...

We choose sine solutions since $G(0) = G(H) = 0$:

$$G(y) = \sin(\sqrt{\lambda}\, y)$$

For BC's: $\sqrt{\lambda} = \frac{n\pi}{H}$, $n = 1, 2, 3, \dots$

$$\Longrightarrow \lambda = \left(\frac{n\pi}{H}\right)^2$$

$$\Longrightarrow \boxed{G(y) = \sin\left(\frac{n^{\cancel{2}}\pi^{\cancel{2}}}{H^{\cancel{2}}}\, y\right)}$$

Details:

Assume $G(y) = e^{\alpha y}$

so $G_{yy} = \alpha^2 e^{\alpha y}$

$\Longrightarrow (\alpha^2 + \lambda) e^{\alpha y} = 0$

$\Longrightarrow \alpha^2 = -\lambda$

$\Longrightarrow \alpha = \pm i \sqrt{\lambda}$

so $G(y) = k_1 \cos(\overset{\sqrt{\lambda}}{\cancel{\lambda}}\, y) \pm k_2 \sin(\overset{\sqrt{\lambda}}{\cancel{\lambda}}\, y)$

Choose k_1, k_2 so $G(y) = \sin(\sqrt{\lambda}\, y)$.

Case 2 (also interesting, less useful): $\lambda < 0$

if $\lambda < 0$ then $G_{yy} = \overbrace{k^2}^{-\lambda \text{ is positive}} G$

has sol'ns $k_1 e^{ky} + k_2 e^{-ky}$

... cannot satisfy BCs!!

Case 3 (Exercise): $\lambda = 0$?

Now, we solve: $F_{xx} = \lambda F$ for special λ that satisfy top & bottom BCs (in $G_{yy} = -\lambda G$ eq$^{\underline{n}}$):

$$F_{xx} = \left(\frac{n\pi}{H}\right)^2 F \quad ; \quad n = 1, 2, 3, \ldots \quad (n \in \mathbb{Z}^+)$$

Solution: $F(x) = A_n e^{\frac{n\pi}{H}x} + B_n e^{-\frac{n\pi}{H}x}$

From left BC, $F(0) = 0 \implies B_n = -A_n$

$$F(x) = A_n\left[e^{\frac{n\pi}{H}x} + e^{-\frac{n\pi}{H}x}\right] = 2A_n \sinh\left(\frac{n\pi}{H}x\right)$$

So, solution is given by:

$$u(x,y) = F(x)\,G(y)$$

$$\implies \boxed{u(x,y) = \cancel{A_0 x} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H}y\right) \sinh\left(\frac{n\pi}{H}x\right)}$$

$A_0 = 0 \ldots$

For $n=0$, $F_{xx} = 0$

$\Rightarrow F(x) = c_1 + c_2 x$

$F(0) = 0 \Rightarrow F(x) = c_2 x$

but A_0 must equal 0 for BCs...

Last Thing (its a doozy!):

We need to solve for A_n to satisfy last BC:

$$u(L,y) = f_3(y) \ldots$$

Apply BC on $x = L$:

$$u(L,y) = f(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left(\frac{n\pi L}{H}\right)$$

$$\int_0^H f(y) \sin\left(\frac{m\pi}{H} y\right) dy = \int_0^H \sum A_n \sin\left(\frac{n\pi}{H} y\right) \underbrace{\sinh\left(\frac{n\pi L}{H}\right)}^{\text{a number...}} \sin\left(\frac{m\pi}{H} y\right) dy$$

orthogonal
if $m \neq n$ ...

$$= \frac{A_n H}{2} \sinh\left(\frac{n\pi L}{H}\right)$$

$$\Longrightarrow \boxed{A_n = \frac{2}{H \sinh(n\pi L/H)} \int_0^H f(y) \sin\left(\frac{n\pi}{H} y\right) dy}$$