Overview of Topics

1. Nonlinear (black body) radiation
2. Heat Eq^n in 2D & 3D
3. 2D Laplace?

Heat Equation in 2D or 3D (or ND ...!)

Same idea of conservation:

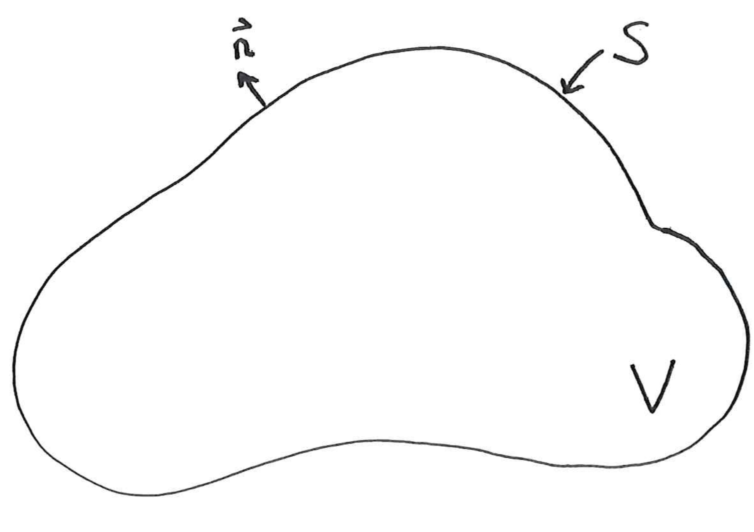
Rate of change of heat energy (in time) = Heat energy flux through boundary of volume + Heat energy generated ~~[illegible]~~ over entire volume

Note: Now we are using a control volume!

(This should be familiar from the derivation of mass conservation from L23 in ME564)

$$\frac{d}{dt}\iiint_V \underbrace{c\rho\, u(\vec{x},t)}_{\text{scalar field}}\, dV = -\oint\!\!\oint_S \underbrace{\vec{q}(\vec{x},t)}_{\text{vector field: } \vec{q}=\nabla u}\cdot\vec{n}\, dS + \iiint_V \underbrace{Q(\vec{x},t)}_{\text{scalar field}}\, dV$$

(single # at each spatial location x)



(continued)

$$\frac{d}{dt}\iiint_V c\rho\, u(\vec{x},t)\,dV = -\oiint_S \vec{q}(\vec{x},t)\cdot\vec{n}\,dS + \iiint_V Q(\vec{x},t)\,dV$$

Now, use Gauss's Divergence Theorem to convert <u>flux</u> <u>integral</u> into volume integral of divergence!

$$\oiint_S \vec{F}\cdot n\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$\left.\frac{d}{dt}\iiint_V c\rho\, u(\vec{x},t)\,dV = -\iiint_V \nabla\cdot\vec{q}(\vec{x},t)\,dV + \iiint_V Q(\vec{x},t)\,dV \quad\right\}\text{All Volume Integrals!}$$

$$\iiint_V \left[c\rho\frac{\partial u}{\partial t} + \nabla\cdot\vec{q} - Q\right]dV = 0$$

Since this is true for <u>all</u> volumes V:

$$\boxed{c\rho\frac{\partial u}{\partial t} + \nabla\cdot\vec{q} - Q = 0}$$

$$\Bigg\Downarrow \quad \text{Fourier's Law of Heat Conduction} \quad \vec{q} = -K\nabla u$$

$$\nabla\cdot\vec{q} = -K\,\nabla\cdot(\nabla u) = -K\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\cdot\begin{bmatrix}\partial u/\partial x\\ \partial u/\partial y\\ \partial u/\partial z\end{bmatrix}$$

$$= -K\left[u_{xx}+u_{yy}+u_{zz}\right]$$

$$= -K\nabla^2 u.$$

$$\boxed{c\rho\frac{\partial u}{\partial t} - K\nabla^2 u - Q = 0}$$

The general heat equation for a system with no sources $(Q=0)$ and constant thermal parameters (c, ρ, K) is:

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2 \nabla^2 u}$$

some positive #.

Equilibrium (Laplace!)

In 2D: $u_t = \alpha^2 (u_{xx} + u_{yy})$

$u_{xx} + u_{yy} = 0$

In 3D: $u_t = \alpha^2 (u_{xx} + u_{yy} + u_{zz})$

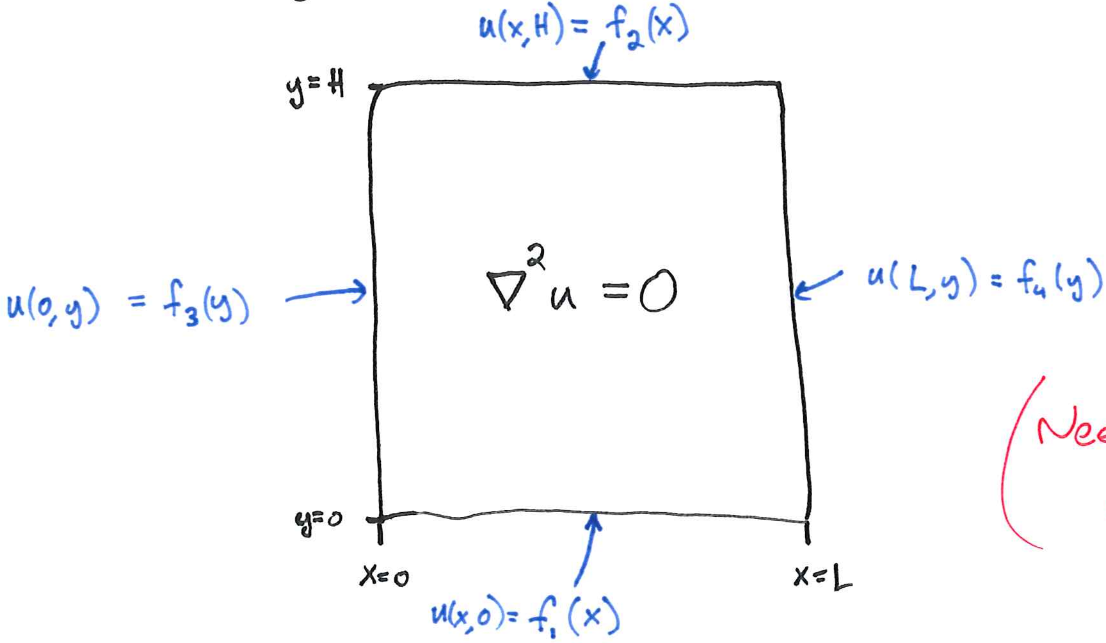
$u_{xx} + u_{yy} + u_{zz} = 0$

In ND: $u_t = \alpha^2 \nabla^2 u$

$\nabla^2 u = 0$

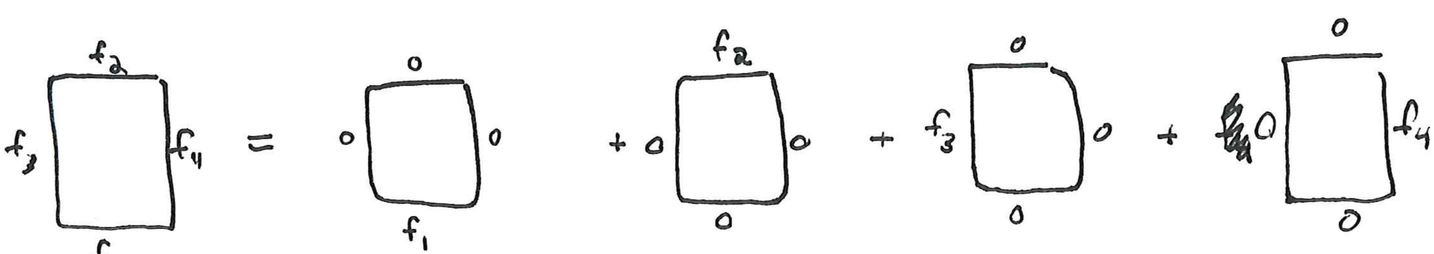
$$u_t = \alpha^2 \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}$$

Lets try Laplaces Equation in 2D:



(Need four BC's on rectangle)

Because $\nabla^2 u = 0$ is <u>linear</u>, we can solve four different systems with simpler BC's and sum the solutions:



$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

Lets work out one of these problems: