L08: Jan 23, 2015

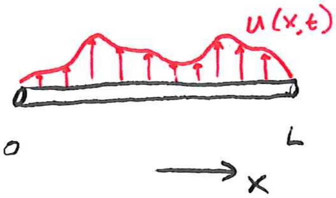
ME565, Winter 2005

Overview of Topics

1. Heat Equation
   a. Linear vs Nonlinear
   b. Derivation
2. Solution to Laplaces Eq^{n}.

Deriving the 1D Heat Equation:

Consider the temperature distribution $u(x,t)$ in a thin metal rod:

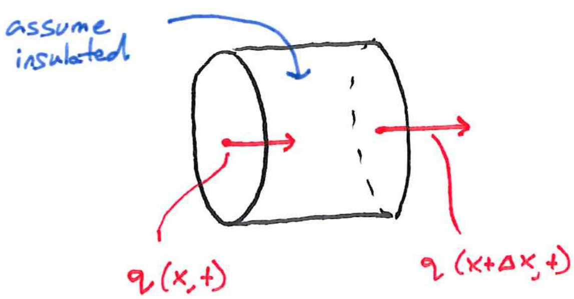


The basic idea is to take "common sense" physics and find a way to write as an equation:

The rate of change of heat energy (in time) = heat flux through boundary (at each point in space) + heat energy generated at each point in space.

$$(*)\qquad \frac{d}{dt}\Big(\overbrace{\underbrace{c(x)}_{\text{specific heat}}\ \underbrace{\rho(x)}_{\text{density}}\ \underbrace{u(x,t)}_{\text{temperature}}}^{\text{heat energy}}\Big) = \underbrace{-\frac{\partial q}{\partial x}}_{\substack{\text{heat flux term}\\ \text{(see below)}}} + \underbrace{Q(x,t)}_{\text{heat sources...}}$$

To understand $-\frac{\partial q}{\partial x}$ term, consider a small section:



$q(x,t)$ is the heat flux from left to right (i.e. thermal energy / time)

if $q(x,t) = q(x+\Delta x,t)$ then no change to heat inside at x!

So we need gradient: $\displaystyle \frac{\partial q}{\partial x} = \lim_{\Delta x \to 0} \frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}$.

Finally, we need an expression for how the heat flux q depends on the temperature.

Enter Fourier (not for the last time...), with some important observations:

1. no heat flux when temperature is constant $(u(x,t) = c)$
2. heat energy flows from higher to lower temperatures
3. more heat flux for larger temp. diff $(\Delta u / \Delta x)$

1-3 $\Longrightarrow$ Fourier's Law of Heat Conduction:

$$\boxed{q(x,t) = -K \frac{\partial u}{\partial x}.}$$

Combining with earlier Eq$^{\underline{n}}$ (*) we recover the heat equation:

$$\boxed{c(x)\rho(x) \frac{\partial u(x,t)}{\partial t} = \underbrace{K \frac{\partial^2 u}{\partial x^2}}_{\text{really } \frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right)} + Q}$$

if thermal conductivity K varies spatially...

Analyzing the 1D Heat Equation:

First, notice that our heat equation is nonlinear, unless we assume constant density and specific heat (and thermal conductivity)

$$\underbrace{c(x)\rho(x)\frac{\partial u(x,t)}{\partial t}}_{\text{Nonlinear!}} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + Q$$

$$c\rho u_t = K u_{xx} + Q \qquad \text{if } c, \rho, K \text{ constant}$$

$\Downarrow$ assume no heat sources $Q=0$

$$\boxed{u_t = \frac{K}{c\rho} u_{xx}} \qquad \text{Linear!}$$

This equation describes the physics mathematically, but we need more information to solve a real problem:

1. Initial Conditions

$u(x,0) = T(x)$ temperature distribution at time $t=0$.

2. Boundary Conditions

What happens at $x=0$ and $x=L$?

A few important boundary conditions (BCs):

1. Fixed temperature:

$u(0,t) = u_B(t)$ (similar for $u(L,t)$)

2. Insulated boundaries:

$u_x(0,t) = 0$

$u_x(L,t) = 0$

Equilibrium Temperature (Laplace's Equation)

In equilibrium, we have $u_t = 0$ (no change),

So our (insulated) heat equation becomes Laplace's Eq^n:

$$0 = u_t = \alpha u_{xx} \implies u_{xx} = 0 \qquad \left(\text{Laplace in 1D: } \nabla^2 u = 0\right)$$

Different Solutions for Different BCs:

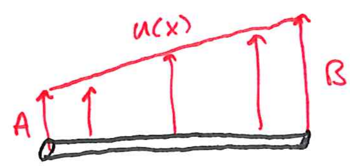
1. Fixed Temperature: $u(0) = A$, $u(L) = B$

$$u_{xx} = 0 \implies u_x = C_1 \implies u(x) = C_1 x + C_2$$

$u(0) = A \implies C_2 = A$

$u(L) = B \implies B = L C_1 + A$

$\implies C_1 = \dfrac{B-A}{L}$



$$\boxed{u(x) = \frac{B-A}{L} x + A}$$

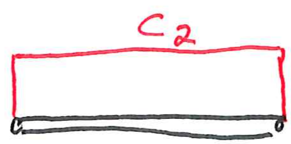
2. Insulated: $u_x(0) = 0$, $u_x(L) = 0$

Still $u(x) = C_1 x + C_2$

but $u_x = C_1$ must $= 0$!

$u(x) = C_2$ a constant!

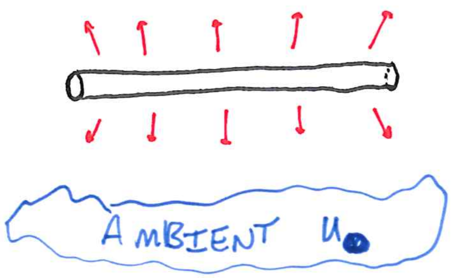
C_2 is not arbitrary!



It is the average temperature of initial distribution $u(x,0) = f(x)$

$$C_2 = \frac{1}{L} \int_0^L u(x,0)\, dx \ldots$$

What if the rod is not insulated?



If the outside temperature is u_0, then there will be heat flux out:

$$u_t = \frac{K}{c\rho} u_{xx} - \mu (u - u_0)$$

At very large temperatures, we have black body radiation given by the Stefan-Boltzmann law

$$u_t = \frac{K}{c\rho} u_{xx} \underbrace{- m (u - u_0)^4}_{\text{nonlinear radiation}}$$