Overview of Topics

1. PDEs
   - types, construction, etc.
2. Canonical PDEs
   - Laplace's Equation
   - Wave Equation
   - Heat Equation

Partial Differential Equations (PDEs)

A partial differential equation is an equation relating multivariate functions and their partial derivatives.

Often the multivariate function "u" is a function of time and space

$$\underbrace{u}_{\text{function}}(\underbrace{x}_{\text{space}},\underbrace{t}_{\text{time}})$$

Example PDE: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (Wave eq$^{\underline{n}}$ more on this later...)

We typically solve the PDE for the unknown function "u", which generally has a physical interpretation

Note: We often denote partial derivatives with subscript: $u_t \triangleq \dfrac{\partial u}{\partial t}$

Canonical Partial Differential Equations:

Linear: No products or cross-terms between $u, u_x, u_{xx}, \ldots$

Second Order: No more than $\underline{2}$ derivatives in any term

Homogeneous: No time dependent forcing or spatial forcing.

I. Wave Equation (hyperbolic)

$$u_{tt} = c^2 \nabla^2 u \qquad\qquad u_{tt} = c^2 u_{xx}$$

II. Heat Equation (parabolic)

$$u_t = \alpha^2 \nabla^2 u \qquad\qquad u_t = \alpha^2 u_{xx}$$

III. Laplace's Equation (elliptic)

$$\underbrace{0 = \nabla^2 u}_{\text{General Vector Form}} \qquad\qquad \underbrace{u_{xx} + u_{yy} = 0}_{\text{Specific Form for 2 Independent Variables.}}$$

Note: In each example, u is a scalar function of multiple variables.

Linear PDEs are built up using <u>Linear Operators</u> "L":

① multiplication by a constant : $L_1(u) = 5u$

② partial derivative : $L_2(u) = \partial u/\partial x$

So $L_1(L_2(u)) = 0 \Rightarrow 5\,\partial u/\partial x = 0$.

Linearity means that $\underline{A}$ $L(u_1+u_2) = L(u_1) + L(u_2)$ for all u_1, u_2

and $\underline{B}$ $L(au_1) = aL(u_1)$ for all u_1 and constant a.

Examples above: $L_1(u_1+u_2) = 5(u_1+u_2) = 5u_1 + 5u_2 = L_1(u_1) + L_1(u_2)$ ✓

$L_1(au) = 5(au) = a\,5u = a L_1(u)$ ✓

$L_2(u_1+u_2) = \frac{\partial}{\partial x}(u_1+u_2) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = L_2(u_1) + L_2(u_2)$ ✓

$L_2(au) = \frac{\partial}{\partial x}(au) = a\frac{\partial u}{\partial x} = aL_2(u)$ ✓

What about ③ addition by constant : $L_3(u) = u + 7$?

$L_3(u_1+u_2) = u_1+u_2+7 \neq L_3(u_1) + L_3(u_2) = u_1+u_2+14$.

So L_3 is <u>NOT</u> linear. L_3 is called an <u>Affine</u> <u>Linear</u> <u>Operator</u>.

We have tons of tools to solve (and understand) <u>linear</u> PDEs.

In general, nonlinear PDEs are much more difficult, although many important physical systems are nonlinear.

Examples of Nonlinearity:

Squaring, cubing, etc: $N(u) = u^2$

$$N(u_1+u_2) = (u_1+u_2)^2 = \underbrace{u_1^2+u_2^2}_{N(u_1)+N(u_2)} + 2u_1u_2 \quad \times$$

$N(u_1)+N(u_2)$ + extra terms!

Products: $N(u) = u \cdot u_x$

$$N(u_1+u_2) = (u_1+u_2)\cdot\frac{\partial}{\partial x}(u_1+u_2) = u_1u_{1x} + u_1u_{2x} + u_2u_{1x} + u_2u_{2x} \quad \times$$

extra terms

$$\neq u_1u_{1x}+u_2u_{2x}$$

Absolute Value: $N(u) = |u|$

$$N(5-7) = |-2| = 2$$

$$N(5)+N(-7) = 5+7 = 12 \quad \times$$

Important Example: Burger's' Equation (canonical example w/ shock waves)

nonlinear convection produces shocks

$$u_t + uu_x = \nu u_{xx}$$ ← viscous diffusion spreads shocks...

$$\left(\vec{u}_t + \vec{u}\cdot\nabla\vec{u} = \nu\nabla^2\vec{u}\right)$$ ← vector form.....