Overview of Topics

1. Inverse Laplace transform (using $\mathbb{C}$ - integrals)

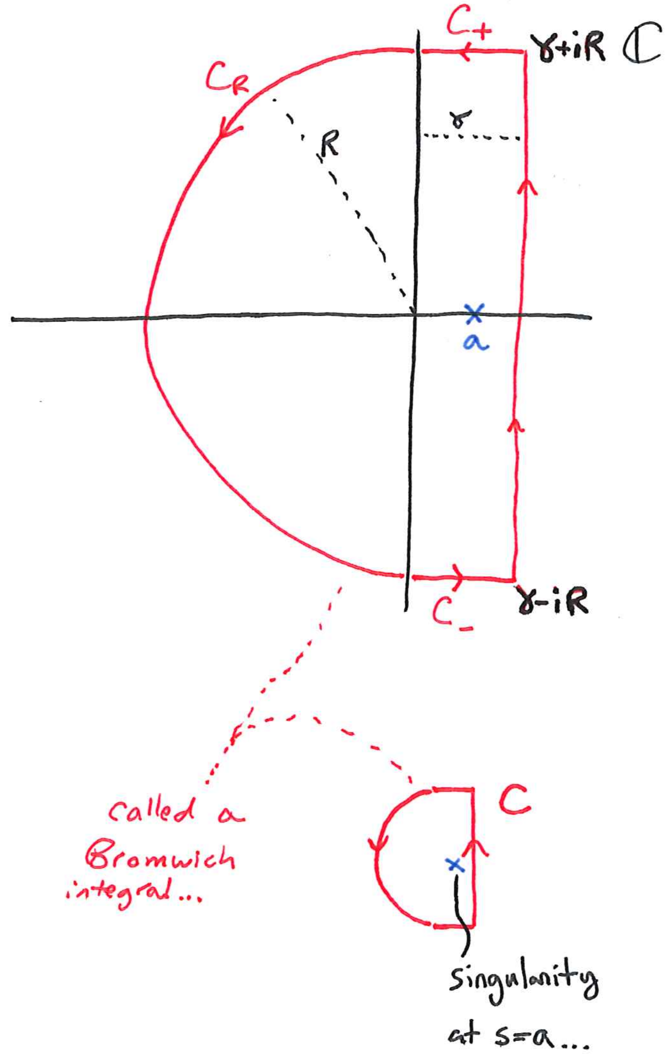
2. More examples of complex integrals.

Example : Inverse Laplace Transform ✱(IMPORTANT)✱

$$\mathcal{L}^{-1}\left\{\hat{f}(s)\right\} \overset{\text{def}^{\underline{n}}}{\triangleq} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \hat{f}(s)e^{st}\,ds \quad ; \quad \begin{array}{l}\gamma > \text{real part of all poles of } \hat{f} \\ t \geq 0\end{array}$$

Specific Example: $\hat{f}(s) = \dfrac{1}{s-a}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \underbrace{\frac{e^{st}}{s-a}}_{\substack{\text{Looks like perfect} \\ \text{application of CIF... make a big contour!}}}ds \quad ; \quad \begin{array}{l}\gamma > a \\ t \geq 0\end{array}$$



By Cauchy Integral Theorem,

$$\frac{1}{2\pi i}\oint_C \frac{e^{st}}{s-a}\,ds = e^{at}$$

So $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \lim\limits_{R\to\infty} \displaystyle\int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{s-a}\,dt = e^{at}$

If we can show that

$$\lim_{R\to\infty} \int_{C_+} + \int_{C_R} + \int_{C_-} = 0 .$$

Big if...

(continued)

$\underline{\underline{\int_{C+}, \int_{C-}}}$: Use ML bound, since length of path is always $L = \gamma$, regardless of $R \to \infty$.

$$\int_{C+} \frac{e^{st}}{s-a} ds = \int_{\gamma+iR}^{iR} \frac{e^{st}}{s-a} ds = \int_{\gamma}^{0} \frac{e^{x+iR}}{x+iR-a} dx \leq ML \quad (L=\gamma).$$

$$M = \max_{x \in [0,\gamma]} \left| \frac{e^{(x+iR)t}}{x+iR-a} \right| \underset{\text{actually } =}{\leq} \frac{e^{\gamma t}}{R}$$

(ie biggest numerator, smallest denominator...)

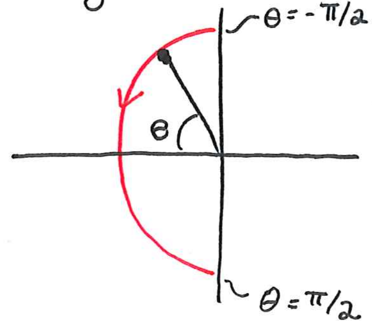
$$\text{So } \int_{C+} \leq \frac{\gamma e^{\gamma t}}{R} \longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

Similar for $\int_{C-}$.

$$\text{So } \int_{C+} = 0 \text{ and } \int_{C-} = 0 \,!$$

(continued)

$\underline{\underline{\int_{C_R}}}$: ML doesnt work for this part... try polar coords...



$$x = -R\cos(\theta), \quad y = R\sin(\theta)$$

$$|e^{st}| = e^{\overbrace{-Rt\cos(\theta)}^{\text{real part... of } st.}}$$

$$|s-a| = |-R\cos(\theta) - a + iR\sin(\theta)| = \sqrt{(R\cos + a)^2 + R^2\sin^2} = \sqrt{R^2 + 2aR\cos + a^2} \geq |R-a|$$

$$\text{So } \left|\int_{C_R} \frac{e^{st}}{s-a} ds\right| \leq \int_{-\pi/2}^{\pi/2} \frac{e^{-Rt\cos\theta}}{|R-a|} \underbrace{R\,d\theta}_{\text{polar coords...}} = \underbrace{\frac{2R}{|R-a|} \int_0^{\pi/2} e^{-Rt\cos(\theta)} d\theta}_{\text{pulled out 2 and changed } \int_{-\pi/2}^{\pi/2} = 2\int_0^{\pi/2} \text{ since even function...}}$$

Need another trick... $\cos(\theta) \geq 1 - 2\theta/\pi$ on $\theta \in [0, \pi/2]$

$$e^{-Rt\cos(\theta)} \leq e^{-Rt(1-2\theta/\pi)} = e^{Rt(2\theta/\pi - 1)}.$$

$$\text{Finally, } \left|\int_{C_R} \frac{e^{st}}{s-a} ds\right| \leq \frac{2R}{|R-a|} \int_0^{\pi/2} e^{Rt\left(\frac{2\theta}{\pi} - 1\right)} d\theta = \frac{2R}{R-a} \frac{\pi}{2Rt} e^{Rt\left(\frac{2\theta}{\pi} - 1\right)}\Big]_0^{\pi/2}$$

$$= \frac{\pi}{|R-a|t}\left(1 - e^{-Rt}\right) \longrightarrow 0 \quad \text{as } R \to \infty.$$

So $\int_{C_R} = 0$.

$$\text{Thus } \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \ldots$$

Example: $\displaystyle\int_0^{2\pi} \sin^2\theta \, d\theta$

Method I (calculus): $\displaystyle\int_0^{2\pi} \sin^2(\theta)\, d\theta = \int_0^{2\pi} \left(\frac{1-\cos(2\theta)}{2}\right) d\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_{\theta=0}^{\theta=2\pi} = \pi.$

Method II (complex): Turn into contour integral.

Set
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

and
$$z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta \implies d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\underbrace{\int_0^{2\pi} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 d\theta}_{\int_0^{2\pi} \sin^2(\theta)\, d\theta} = \oint_C \frac{(z - 1/z)^2}{-4} \frac{dz}{iz}$$

$$= \frac{i}{4} \oint_C \frac{z^2 - 2 + 1/z^2}{z} dz$$

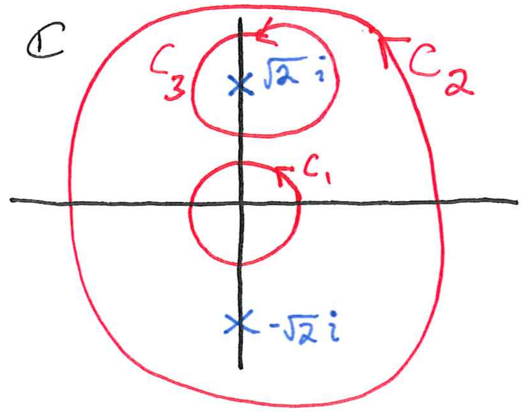
$$= \frac{i}{4} \oint_C \left(z - \frac{2}{z} + \frac{1}{z^3}\right) dz$$

only term that doesn't $\oint_C$ to 0...

$$= -\frac{i}{2} \oint_C \frac{1}{z} dz = -\frac{i}{2} \cdot 2\pi i = \underline{\underline{\pi}}. \; !$$

Example: We can solve hard integrals using CIF

$$\int_C \frac{\sin(z)}{z(z^2+2)}\,dz$$



for $C_1 = |z| = 1$

$C_2 = |z| = 2$

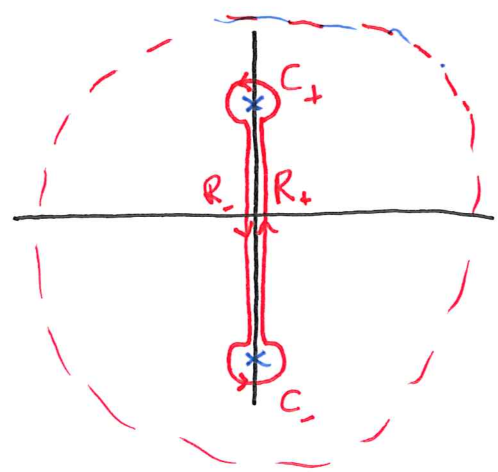
and $C_3 = |z - \sqrt{2}\,i| = 1$

Note: $\dfrac{\sin(z)}{z}$ analytic everywhere (verify!) so no pole at $z=0$.

Singularities at $z = \pm\sqrt{2}\,i$

Contour C_1: No sing. inside C_1, so $\int_{C_1} f(z)\,dz = \int_{C_1} \frac{\sin z}{z(z^2+2)}\,dz = 0$.

Contour C_2: Deform contour...



$$\int_{C_2} \frac{\sin(z)}{z(z+\sqrt{2}\,i)(z-\sqrt{2}\,i)}\,dz$$

$$= \int_{C_+} + \int_{C_-} + \underbrace{\int_{R_-} + \int_{R_+}}_{\substack{\text{cancel} \\ =0}}$$

$$= \int_{C_+} \left(\frac{\sin(z)}{z(z+\sqrt{2}\,i)}\right) \cdot \frac{dz}{(z-\sqrt{2}\,i)} + \int_{C_-} \frac{\sin(z)}{z(z-\sqrt{2}\,i)} \cdot \frac{dz}{z+\sqrt{2}\,i}$$

$f(z)$ $/(z-a)$

analytic near $a = \sqrt{2}\,i$

$$= \underbrace{\frac{\sin(\sqrt{2}\,i)}{\sqrt{2}\,i(2\sqrt{2}\,i)} \cdot 2\pi i}_{\int_{C_+} \text{ by CIF}} + \underbrace{\frac{\sin(-\sqrt{2}\,i)}{-\sqrt{2}\,i(-2\sqrt{2}\,i)} \cdot 2\pi i}_{\int_{C_-} \text{ by CIF}} = 0$$

cancel!