Overview of Topics

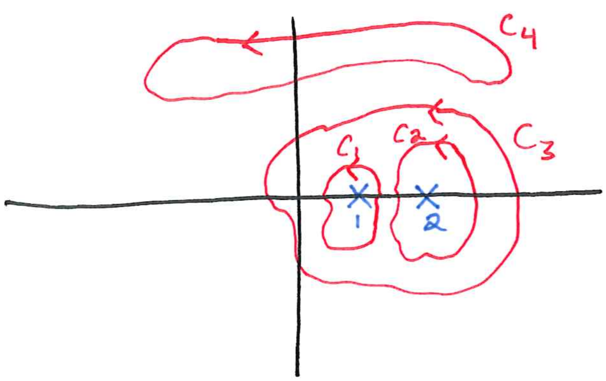
1. ML bound
2. Examples of Cauchy Integral formula

Example: $f(z) = \dfrac{1}{z^2-3z+2} = \dfrac{1}{(z-1)(z-2)}$ ... poles at $z=1,2$

Note: $\dfrac{1}{z-1}$ is analytic near $z=2$

and $\dfrac{1}{z-2}$ is analytic near $z=1$, so CIF applies: $\displaystyle\int_C \frac{f(z)\,dz}{z-a} = 2\pi i f(a)$

different f from above



$$\int_{C_1} f(z)\,dz = \int_{C_1} \frac{1}{(z-2)} \cdot \frac{dz}{(z-1)} = \left[\frac{1}{z-2}\right]_{z=1} \cdot 2\pi i = \underline{\underline{-2\pi i}}$$

$$\int_{C_2} f(z)\,dz = \int_{C_2} \frac{1}{(z-1)} \frac{dz}{z-2} = \left[\frac{1}{z-1}\right]_{z=2} \cdot 2\pi i = \underline{\underline{2\pi i}}$$

$\displaystyle\int_{C_3} = 0$ (poles cancel!)

$\displaystyle\int_{C_4} = 0$ (analytic!)

Another Useful Formula

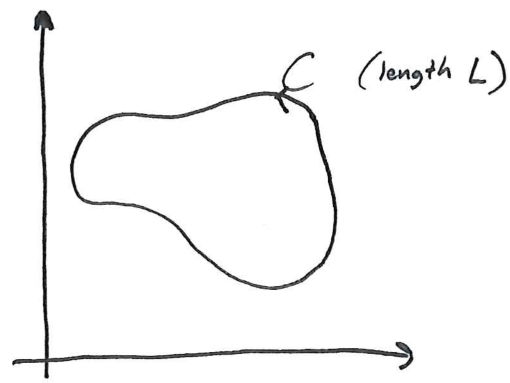
ML Bound: If $|f| \leq M$ on C

and $\int_C ds = L$,

then $\left| \int_C f(z)\,dz \right| \leq ML$

M times L is worst-case scenario bound...

... makes sense...

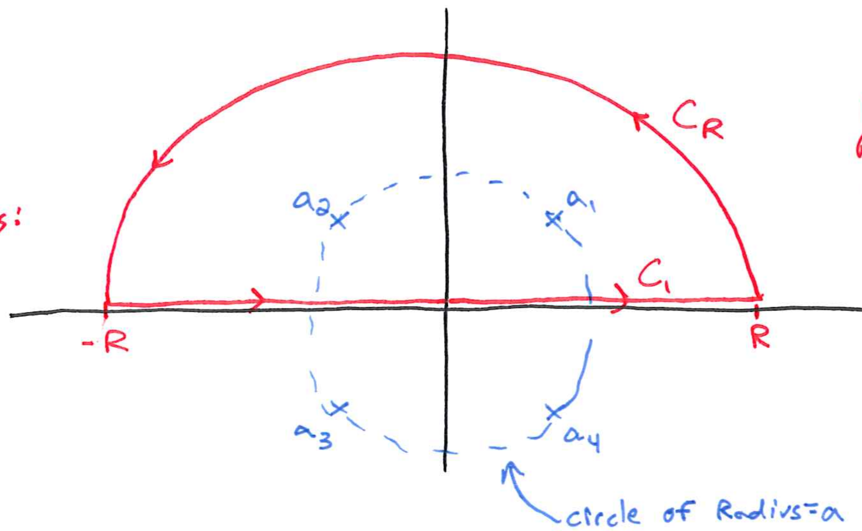


We can also solve $\int_{-\infty}^{\infty} f(x)\,dx$ (real-valued f) using Complex...

Example: $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}$ ($f(x)$ is even!)

Turn this into contour integral in $\mathbb{C}$:

$C = C_1 + C_R$ is closed contour containing two poles: a_1, a_2.



Note: we will take $\lim_{R\to\infty}$, and C_1 will become our integral above.

$$\int_C \frac{dz}{\boxed{z^4+a^4}} = \int_{C_1+C_R} = \int_{-R}^{R}\frac{dx}{x^4+a^4} + \int_{C_R}\frac{dz}{z^4+a^4}$$

four poles at
$z^4 = -a^4 \Rightarrow z = a\sqrt[4]{-1}$
$= \dfrac{\pm a \pm ia}{\sqrt{2}}$

$(z^4+a^4) = (z-a_1)(z-a_2)(z-a_3)(z-a_4)$

$$= 2\pi i\left[\underbrace{\frac{1}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}}_{\text{CIF for pole at } a_1} + \underbrace{\frac{1}{(a_2-a_1)(a_2-a_3)(a_2-a_4)}}_{\text{CIF for pole at } a_2}\right]$$

$$= 2\pi i\left[\frac{1}{\left(\frac{a+ia-(-a+ia)}{\sqrt{2}}\right)\left(\frac{a+ia-(-a-ia)}{\sqrt{2}}\right)\left(\frac{a+ia-(a-ia)}{\sqrt{2}}\right)}\right.$$
$$\left.+\frac{1}{\left(\frac{-a+ia-(a+ia)}{\sqrt{2}}\right)\left(\frac{-a+ia-(-a-ia)}{\sqrt{2}}\right)\left(\frac{-a+ia-(a-ia)}{\sqrt{2}}\right)}\right]$$

$$\oint_C \frac{dz}{z^4+a^4} = 2\pi i\left[\frac{1}{2\sqrt{2}a^3 i(1+i)} + \frac{1}{2\sqrt{2}a^3 i(1-i)}\right] = \frac{2\pi i}{2\sqrt{2}a^3 i} = \frac{\pi}{\sqrt{2}a^3}$$

(continued...)

Finally, we will show $\lim_{R \to \infty} \oint_{C_R} \frac{dz}{z^4+a^4} = 0$

So $\lim_{R \to \infty} \oint_{C} \frac{dz}{z^4+a^4} = \boxed{\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{\sqrt{2}\, a^3}}$.

We will use the ML bound...

$$\left| \int_{C_R} \frac{dz}{z^4+a^4} \right| \leq ML$$

$L = \pi R$

$M = \max_{z \in C_R} \left| \frac{1}{z^4+a^4} \right| \leq \frac{1}{R^4}$

$$\leq \frac{\pi}{R^3}$$

$\lim_{R \to \infty} ML = 0$. so $\left| \int_{C_R} \cdots \right| \leq 0 \implies \underline{= 0}$.