All polynomials and convergent power series are analytic (within Radius of convergence...)

Functions with singularities are not analytic at singularity.

Ex. $(z-a)^n$ is not analytic for $n = -1, -2, -3, \dots$

However, integral has nice properties.

integers $n = 1, 2, 3 \dots$

Case 1
$n = 1, 2, 3, \dots$
$n = -2, -3, -4, \dots$

$$\int_C (z-a)^n \, dz = \left. \frac{(z-a)^{n+1}}{n+1} \right|_{z_0}^{z_0} = 0 \quad \text{for} \quad \underbrace{n > 0}_{} \ \text{(analytic polynomial)}, \quad n = -2, -3, -4 \dots \ \left(\begin{matrix}\text{not analytic} \\ \text{but simple } \int\end{matrix}\right)$$

Case 2
$n = -1$

$$\int_C (z-a)^{-1} \, dz = \Big[\operatorname{Log}(z-a) \Big]_{z_1}^{z_2} \ \dots \neq 0 \text{ since Log is multi-valued}$$

Two approaches: (I) Deform curve C into a circle of radius R.

So $z - a = Re^{i\theta}$ and $dz = iRe^{i\theta} d\theta$ $\left(\frac{\partial z}{\partial R} = 0 \right)$

$$\int_C (z-a)^{-1} dz = \int_{|z|=R,\ \theta=0}^{\theta = 2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} \, d\theta = \int_0^{2\pi} i \, d\theta = 2\pi i$$

(II) $\Big[\log(R) + i(\theta_p + 2\pi n) \Big]_0^{2\pi} = 2\pi i$

$$\boxed{\int_C (z-a)^n \, dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}}$$

Pivotal Result in Complex Analysis

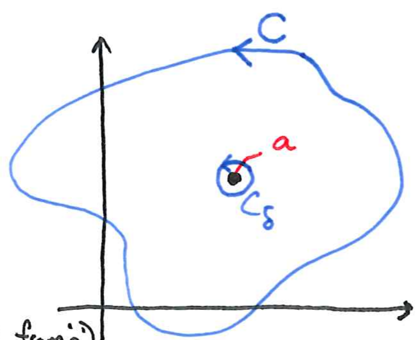
The Cauchy Integral Formula

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' inside C, then:

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)}$$

Simple to derive: $\displaystyle\int_C \frac{f(z)}{z-a}\,dz = \underbrace{\int_C \frac{f(a)}{z-a}\,dz}_{I_1} + \underbrace{\int_C \frac{f(z)-f(a)}{z-a}\,dz}_{I_2}$

$$I_1 = f(a)\int_C \frac{1}{z-a}\,dz = 2\pi i f(a)$$



To show that $I_2 = 0$,

we deform C to C_δ, a circle of radius δ (we can do since analytic away from 'a')

Since $f(z)$ analytic around 'a', can choose δ s.t. $|f(z)-f(a)| < \varepsilon$ on C_δ (can choose such a δ for any $\varepsilon > 0$ by standard Calculus).

$$I_2 = \int_{C_\delta} \frac{f(z)-f(a)}{z-a}\,dz \implies |I_2| \leq \int_{C_\delta} \frac{\varepsilon}{z-a}\,dz = 2\pi i \varepsilon$$

True for all $\varepsilon > 0$ so let $\varepsilon \to 0$ and $I_2 = 0$ must hold!

$$\implies \int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).$$

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clear all, close all, clc

% f(z) = 1/z

N = 1000;
I = 0;
dTheta = (2*pi/N);
for k=1:N
    Theta = 2*pi*k/N;
    z = exp(i*Theta);
    dz = i*exp(i*Theta)*dTheta;
%     I = I + dTheta*(1/(z));
    I = I + dz*(exp(z)/z);
end

I
```