Two Important Theorems for Analytic Functions:

Theorem I: A function $f(z) = u + iv$ that is

- single-valued, and
- has continuous first partials: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

in a domain $\mathcal{D} \subseteq \mathbb{C}$ is analytic in $\mathcal{D}$

if and only if (iff) the Cauchy-Riemann conditions hold at every point $z \in \mathcal{D}$.

Theorem II: [Goursat Theorem] If $f(z)$ is analytic at z, then $f(z)$ has continuous derivatives of all orders.

Major Implication: $f(z)$ is analytic at z_0 iff its Taylor series converges to the function in a neighborhood of z_0.

Analytic Functions, Laplace's Equation, and Harmonic Functions

If a function $f(z) = u(x,y) + iv(x,y)$ is analytic

and both u and v are twice differentiable, then:

CR1 $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \xrightarrow{\partial/\partial x} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$

CR2 $\quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \xrightarrow{\partial/\partial y} \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y}$

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} \\ \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y} \end{array}\right\} \quad \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$$\nabla^2 u = 0$$

Similarly,

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \xrightarrow{\partial/\partial x} \dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{\partial^2 v}{\partial x^2} \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \xrightarrow{\partial/\partial y} \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y^2} \end{array}\right\} \Longrightarrow \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

$$\nabla^2 v = 0$$

The real and imaginary parts of an analytic function satisfy Laplace's equation.

They are called <u>harmonic functions</u>.

Cauchy-Riemann Conditions in Polar coords:

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

or, using $u_r = \partial u/\partial r$ and $u_\theta = \partial u/\partial \theta$:

$$\boxed{u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = \frac{-1}{r} u_\theta}$$

Useful to verify that $\mathrm{Log}(z)$ is analytic away from $z = 0$ (which is a very interesting point!)

$$\mathrm{Log}(z) = \underbrace{\log(r)}_{u(r,\theta)} + i\underbrace{(\theta_0 + 2\pi k)}_{v(r,\theta)}$$

Clearly $u_\theta = v_r = 0$ and $u_r = \frac{1}{r}$, $v_\theta = 1$

CR1: $u_r = \frac{1}{r} v_\theta$ ✓

CR2: $v_r = -\frac{1}{r} u_\theta$ ✓ (this cond^n true for $r > 0$).

Integrals in the Complex Plane:

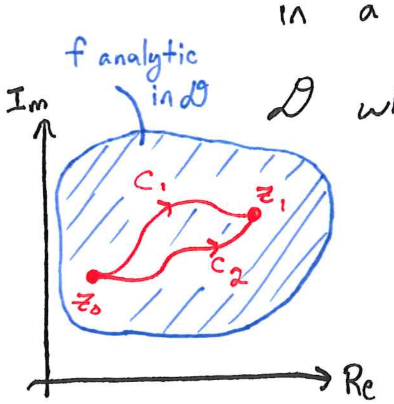
Given a function $f(z) = u(x,y) + iv(x,y)$

then $$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C \left[(u\,dx - v\,dy) + i(v\,dx + u\,dy)\right]$$

Theorem: If C is a curve joining z_0 and z_1, in a closed, simply connected region (i.e., no holes) $\mathcal{D}$ where $f(z)$ is analytic, then

$$\int_C f(z)\,dz$$ is independent of the path.



udx-vdy exact just means that they define a conservative vector field given by the gradient of F...

$$u\,dx - v\,dy = \begin{bmatrix} \partial F/\partial x \\ \partial F/\partial y \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$$

If $u\,dx - v\,dy$ and $v\,dx + u\,dy$ are exact differentials

i.e. $u = F_x$, $v = -F_y$; $v = G_x$, $u = G_y$ $\left(\text{so that } \int_C f\,dz = F + iG\right)$

then u & v satisfy the Cauchy-Riemann conditions,

and f is analytic!

If f is analytic in $\mathcal{D}$ and $z_0, z_1 \in \mathcal{D}$

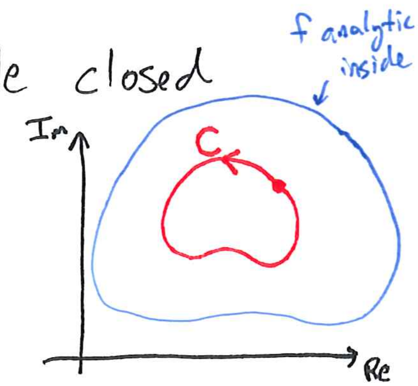
then $$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0).$$

(and F is analytic, too)

Fundamental Theorem of Complex Integral Calculus.

Cauchy-Goursat Integral Theorem (Important)

If $f(z)$ is analytic inside a simple closed curve $C \overset{\text{subset}}{\subset} \mathbb{C}$, then $\int_C f(z)\,dz = 0$.



[Convention: traverse C counter-clockwise ...]

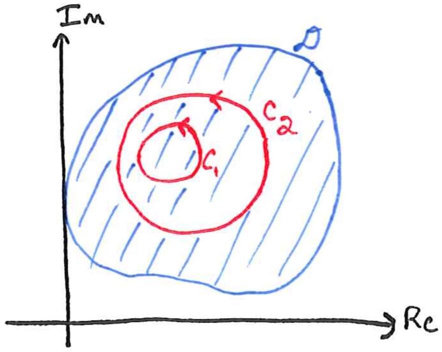
Both real-valued integrals

Proof: $$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

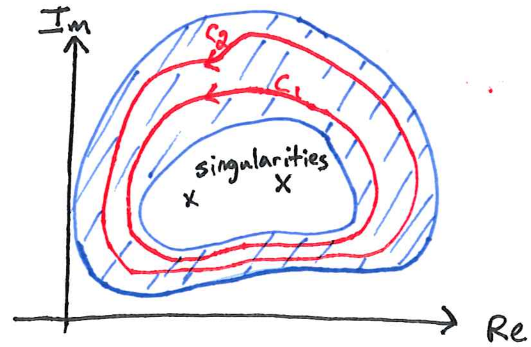
By Green's Theorem $\Longrightarrow$ $$= \iint_S \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{=0 \text{ by CR}} dx\,dy + i \iint_S \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0 \text{ by CR}} dx\,dy$$

$$= 0.$$

Inside an analytic region $\mathcal{D}$, we may deform contours continuously without changing the integral.



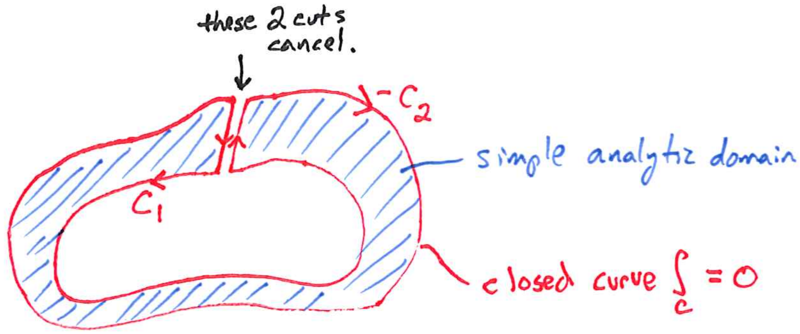
OR EVEN



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \quad (=0)$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \quad (\neq 0)$$

Geometric Sketch:



$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\Longrightarrow \int_{C_1} f(z)\,dz = -\int_{-C_2} f(z)\,dz = \int_{+C_2} f(z)\,dz .$$