Overview of Topics

1. Roots of Unity
2. Branch cuts (optional)
3. Analytic functions and the Cauchy-Riemann Conditions.

Complex numbers in science fiction:

Consider relativistic mass dialation for very fast particle with velocity v and rest mass m_0 (mass if $v=0$):

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad c = \text{speed of light}$$

Usually we argue that particles cannot go faster than the speed of light because they would have ∞-mass as they pass through $v=c$ (divide by zero).

However, what about particles that always travel faster than light?

If $v > c$ (always) then mass is imaginary!

These partizles are called Tachyons, and it has been proposed (dubiously) that neutrinos may tachyons...

"Oh no! Tachyons are flooding the warp core!"

Power function z^a may be rewritten using "exp" and "log":

$$z = e^{\mathrm{Log}(z)} \implies z^a = \left(e^{\mathrm{Log}(z)}\right)^a = e^{a\,\mathrm{Log}(z)}$$

This formula, $\boxed{z^a = e^{a \log(z)}}$ is valid (and useful!)

for <u>any</u> real or complex-valued power 'a'.

<u>Example:</u> Take a rational $a = m/n \in \mathbb{Q} \subset \mathbb{R}$ with $m, n \in \mathbb{Z}$

"an element of" "a subset of"

"set of all rational numbers" "the set of all real numbers" "set of all integers" from German: Zahlen: numbers

$$z^{m/n} = e^{m/n\left(\log|z| + i(\theta_p + 2\pi k)\right)}$$

$$= e^{\frac{m}{n}\log(R)}\, e^{\frac{m}{n} i\theta_p}\, e^{\frac{m}{n} i\, 2\pi k}$$

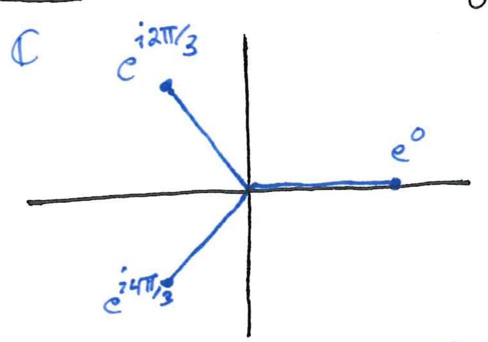
Note: $R = |z| = \sqrt{x^2 + y^2}$

$$\Longrightarrow \boxed{z^{m/n} = R^{m/n} e^{i(m/n)(\theta_p + 2\pi k)}}$$

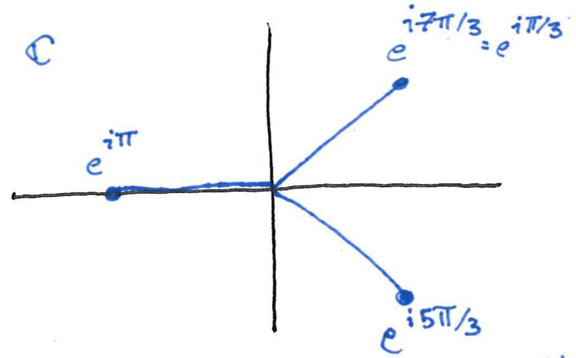
This has n unique values for $k = 0, 1, 2, \ldots, n-1$

i.e. $\frac{m}{n}\left[\theta_p + 2\pi n\right] = \frac{m}{n}\theta_p + 2\pi m$, which is same as $\frac{m}{n}\theta_p$ for $k=0$.

<u>Example:</u> Roots of unity $\sqrt[n]{1} = 1^{1/n} = e^{i 2\pi k/n}$ for $k = 0, 1, 2, \ldots, n-1$.



Three values for $\sqrt[3]{1} = 1^{1/3}$



Three values for $\sqrt[3]{-1} = (-1)^{1/3}$

also works for 'a' irrational (∞-many values) or complex...

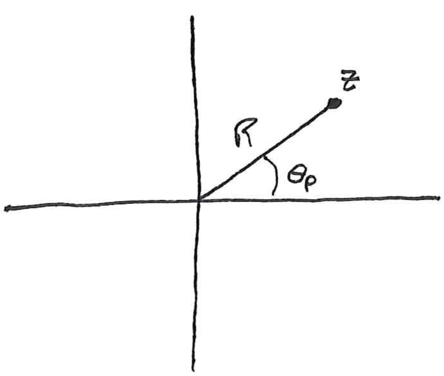
Branch Points & Cuts (multivalued functions):

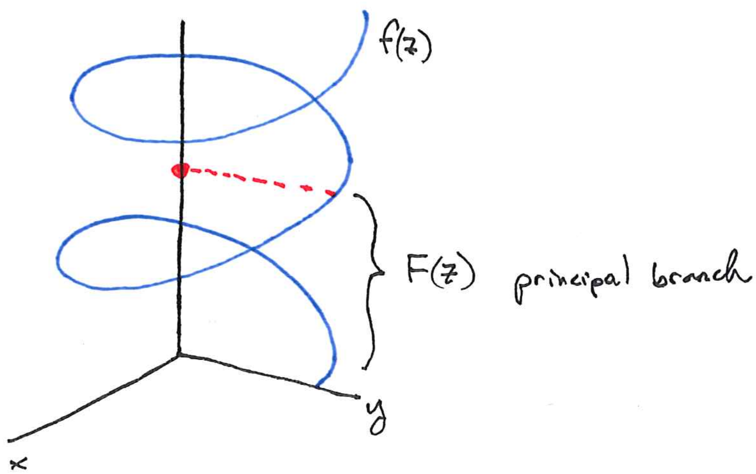
A <u>branch</u> of a multivalued function $f(z)$ is a single-valued analytic function $F(z)$ on a region $\mathcal{D} \subset \mathbb{C}$ that coincides w/ $f(z)$ on one branch.

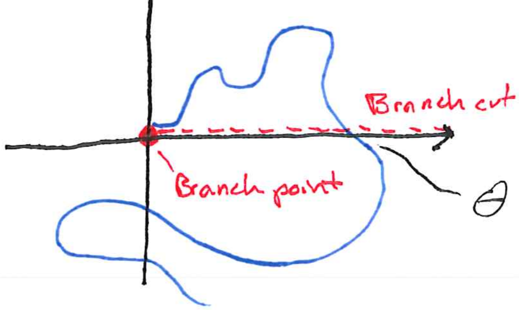
A <u>branch cut</u> $\mathcal{B} \subset \mathbb{C}$ is a curve that bounds the region $\mathcal{D}$. The points $z \in \mathcal{B}$ are singular, meaning that $F(z)$ jumps values on either size of $\mathcal{B}$.

A branch point is a point common to all branch cuts.

<u>Example</u>: $\mathrm{Log}(z) = \log|z| + i\theta_p$ is the principal branch of $\mathrm{Log}(z)$







θ jumps by 2π when branch cut is crossed.

Analytic Functions:

We would like to be able to do Calculus in the complex plane. However, some functions are better behaved than others.

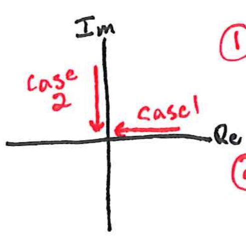
A function is analytic in a domain $\mathcal{D}$ if $f(z)$ is single-valued and has a finite derivative $f'(z)$ for all $z \in \mathcal{D}$.

Example of weird function that is not analytic:

$$f(z) = \bar{z} := x - iy.$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \; . \; = \frac{\Delta x - i\Delta y}{\Delta x + i \Delta y}$$

Let $\Delta z = \Delta x + i\Delta y$ so $\overline{\Delta z} = \Delta x - i\Delta y$.



① Approach $\Delta z = 0$ from real axis (i.e. $\Delta y = 0$): $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = +1$

② Approach $\Delta z = 0$ from imaginary axis (i.e. $\Delta x = 0$): $\displaystyle\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$

Different based on direction!!

So $f'(z)$ is not a (single) finite value, and hence $f(z) = \bar{z}$ is not analytic.

At the least, for a function to be analytic, the derivative must be the same from the two paths taken on the real and imaginary axes.

$$f(z) = u(x,y) + iv(x,y) \qquad \text{where } x,y \text{ are from } z = x+iy$$

so $$\frac{df}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

real-valued partial derivatives

① Approach on Real axis ($\Delta y = 0$): $$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

② Approach on Imaginary axis ($\Delta x = 0$): $$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$\frac{1}{i} = -i$ $\qquad i = e^{i\pi/2} \qquad i^{-1} = e^{-i\pi/2} = -i$

A necessary condition for df/dz to exist is for ① = ②.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann Conditions (CR)

It turns out that the CR conditions are both necessary and sufficient as long as all partials are continuous.

Cauchy Riemann Conditions