Overview of Topics

① Complex numbers & the complex plane $\mathbb{C}$

② Complex functions
(i.e. functions of complex variable $z \in \mathbb{C}$)

- z^n
- Taylor series
- e^z
- trigonometric (sin, cos), hyperbolic (cosh, sinh, tanh)
- Logarithm : $\log(z)$

Complex Variables:

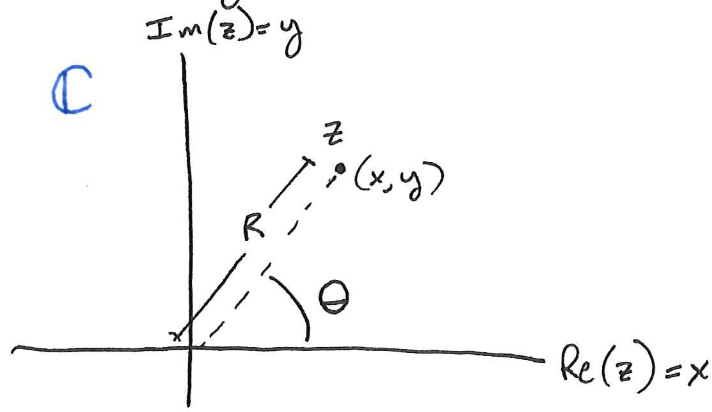
We need complex numbers for solutions to polynomial equations:

$$x^2 + 1 = 0 \implies x = \pm\sqrt{-1}$$

Define: $i = \sqrt{-1}$

Any complex number z may be written as real + imaginary part:

$$z = x + iy$$



Polar coordinates

$$z = R\left[\cos(\theta) + i\sin(\theta)\right]$$

$$x = R\cos(\theta)$$
$$y = R\sin(\theta)$$

$$z = Re^{i\theta}$$

recall the Taylor series for $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Addition & Subtraction

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

Multiplication & Division

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

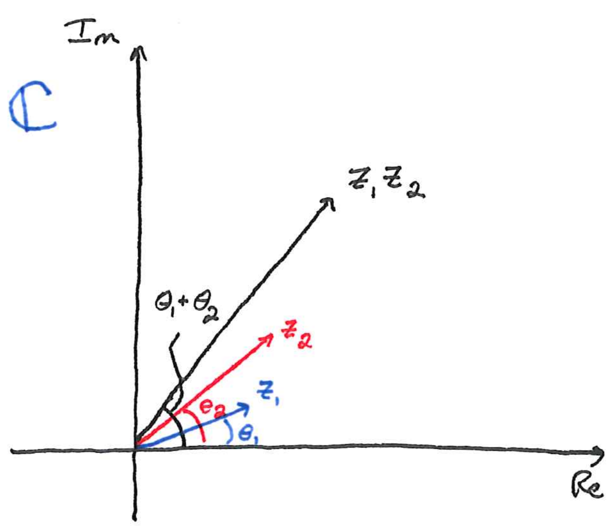
$$z_1 / z_2 = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

Note: $\bar{z} = x - iy$ is the complex conjugate ...

Easier in Polar Coordinates

$$z_1 z_2 = R_1 e^{i\theta_1} R_2 e^{i\theta_2} = R_1 R_2 e^{i(\theta_1 + \theta_2)}$$

$$z_1 / z_2 = \frac{R_1}{R_2} e^{i(\theta_1 - \theta_2)}$$



Functions of z:

$$z^n = \left[R(\cos\theta + i\sin\theta)\right]^n = R^n(\cos\theta + i\sin\theta)^n$$ (Power function)

$$f(z) = \sum_{k=0}^{n} a_k z^k$$ (Power series)

$$f(z) = \frac{\sum_{k=0}^{n} a_k z^k}{\sum_{j=0}^{m} b_j z^j}$$ (Rational function)

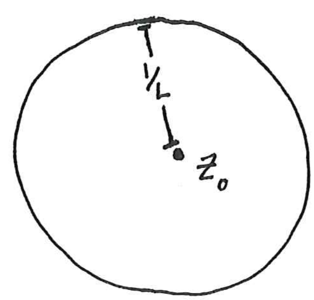
$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$ (Taylor Series)

Taylor series is only convergent if

$$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \text{ exists.}$$

Same in regular calculus!

Then f converges for $|z - z_0| < \frac{1}{L}$ (Radius of convergence)



Exponential Function:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Check convergence:

$$L = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\underbrace{\frac{1}{(k+1)!}}_{a_{k+1}}\cdot\underbrace{\frac{k!}{1}}_{1/a_k}\right| = \lim_{k\to\infty}\left|\frac{1}{k+1}\right| = 0$$

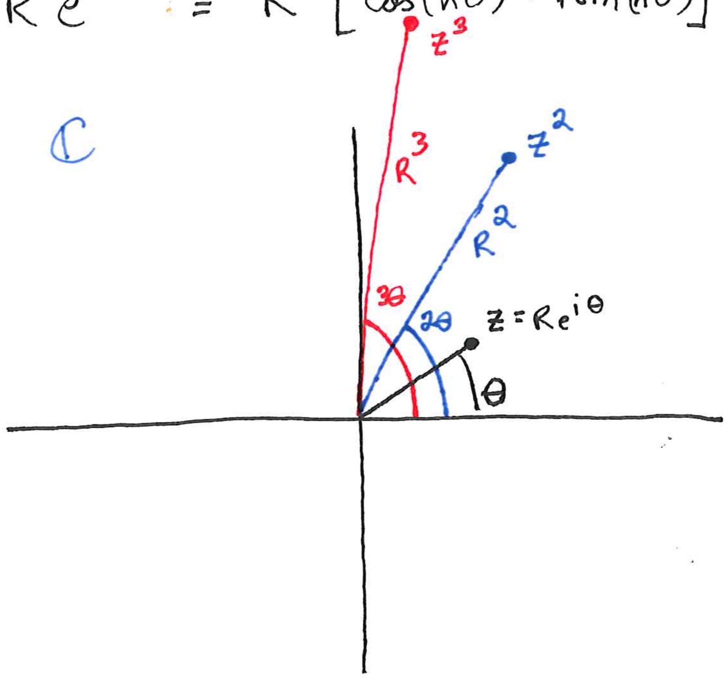
$L \longrightarrow 0$

Thus $L=0$ and e^z has an infinite radius of convergence!

Revisit power function z^n:

$$z = R[\cos\theta + i\sin\theta] = Re^{i\theta} \qquad \text{(Euler's formula)}$$

$$z^n = R^n e^{in\theta} = R^n[\cos(n\theta) + i\sin(n\theta)] \qquad \text{(De Moivre's formula)}$$



Hyperbolic Functions:

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$ Hyperbolic sine "sinch"

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$ Hyperbolic cosine "cosh"

$$\tanh(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}$$ Hyperbolic tangent "tanch"

Trigonometric Functions:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$ Sine

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$ Cosine

Identities:

$$i\sin(z) = \sinh(iz) \qquad \sin(iz) = i\sinh(z)$$

$$\cos(z) = \cosh(iz) \qquad \cos(iz) = \cosh(z)$$

$$e^{z} = \cosh(z) + \sinh(z)$$

$$e^{z} = \cos(iz) - i\sin(iz).$$

Complex Logarithm:

Let $z = x + iy$

$w = u + iv$

and define $w = \mathrm{Log}(z) \Longleftrightarrow z = e^w$ (Log and exp are inverse pair)

$$x + iy = e^u e^{iv} = e^u\left[\cos(v) + i\sin(v)\right]$$

$$\Rightarrow \begin{cases} x = e^u \cos(v) \\ y = e^u \sin(v) \end{cases} \Rightarrow x^2 + y^2 = e^{2u}\left(\cos^2(v) + \sin^2(v)\right)$$

$$\Rightarrow e^u = \sqrt{x^2+y^2} = |z|$$

$$\boxed{u = \log|z|}$$ real-valued logarithm.

Since $x = R\cos(\theta)$, $y = R\sin(\theta)$, we have

$$\boxed{v = \theta = \angle z}$$ angle of z

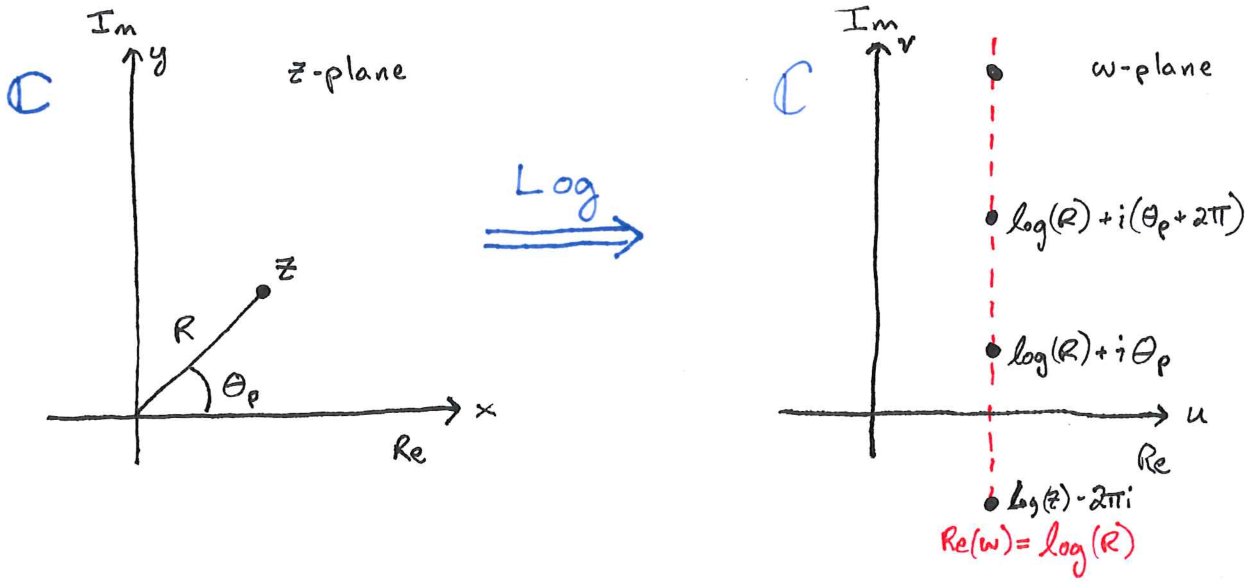
So $w = u + iv = \mathrm{Log}(z) \Rightarrow \mathrm{Log}(z) = \log|z| + i\theta$ $\left(\theta = \angle z = \arg(z)\right)$

However $\theta + 2n\pi$ also works as an angle... pick $\theta_p \in [0, 2\pi)$

$$\boxed{\mathrm{Log}(z) = \log|z| + i(\theta_p + 2n\pi)}$$ for all integers $n \in \mathbb{Z}$

for $n = 0$, we call this the principal value

$\mathrm{Log}(z)$ has ∞ many values!



$\mathbb{C}$
Im
y
z-plane
R
z
θ_p
x
Re
Log
$\mathbb{C}$
Im
v
w-plane
$\log(R) + i(\theta_p + 2\pi)$
$\log(R) + i\theta_p$
u
Re
$\text{Log}(z) - 2\pi i$
$\text{Re}(w) = \log(R)$