
Exercise 5–1: Show the following:

(a) $\nabla \times (\nabla \phi) = 0$ for any potential field ϕ.

(b) $\nabla^2 \phi = \nabla \cdot (\nabla \phi)$ for any potential field ϕ.

(c) $\nabla \cdot (\nabla \phi \times \nabla \psi) = 0$ for all potential fields ϕ and ψ.

Exercise 5–2: Show the following:

(a) $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

(b) $\nabla \cdot (\nabla \times \mathbf{f}) = 0$ for any vector field $\mathbf{f}$.

Exercise 5–3: Here we will compute using the directional derivative $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v}$.

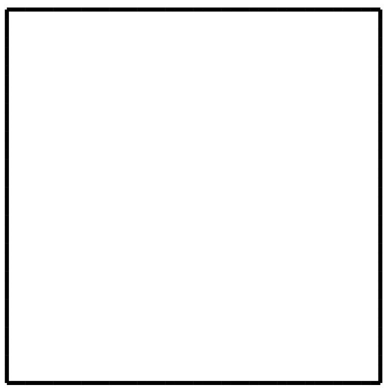
(a) Compute the directional derivative with $f(x, y) = \sin(x) y^2$ and $\mathbf{v} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$.

(b) You are goat climbing a mountain, where the altitude is given by the function $f(x, y) = 20 - x^4 - y^2$. To impress the humans below, you want to choose the steepest path up the mountain. You are at the point $(1, 2, 15)$. What direction should you go in to climb the steepest path?

(c) You are a cold mosquito flying around a room. The temperature is given by $T(x, y, z) = \cos(x) \cos(y) \sin(z)$, and you are at the point $(2, 1, 1)$. What direction should you fly to warm up?

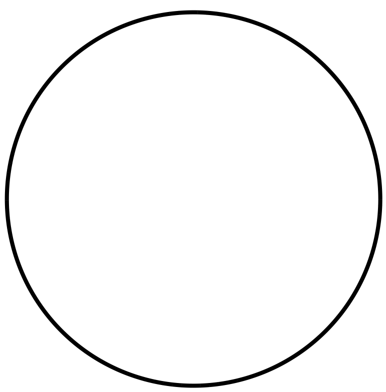
Exercise 5–4:

(a) Compute the area of a square of length 2 centered at the origin using Stokes' theorem:



Square of length 2

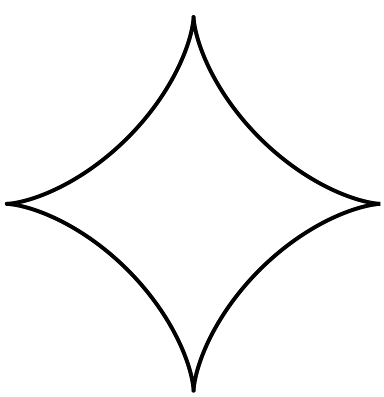
(b) Compute the area of a circle with radius 1 centered at the origin using Stokes' theorem:



Circle: $x^2 + y^2 = 1$

$$x = \cos(\theta)$$
$$y = \sin(\theta)$$

(c) Compute the area of a hypocycloid using Stokes' theorem (Hint: You can compare your answer to the derivation in the notes):



Hypocycloid: $x^{2/3} + y^{2/3} = 1$

$$x = \cos^3(\theta)$$
$$y = \sin^3(\theta)$$