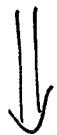


## Overview of Topics

- ① Potential flow & Stream function
- ② Examples

① Start w/ PDE (Laplace's Equation)



② Get vector field  $\vec{v}(x, y) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

③ Use  $\vec{v}$  as right hand side for particle trajectory

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix}$$

$$\underline{\Phi}(z) = \underbrace{\varphi(x,y)}_{\text{potential}} + i \underbrace{\psi(x,y)}_{\text{stream function}}$$

$$\vec{V} = \nabla \varphi = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{bmatrix}$$

$$z = x + iy ; \text{ We want } \nabla^2 \varphi = 0 \text{ and } \nabla^2 \psi = 0$$

$\underline{\Phi}(z) = z^n$  is particularly useful.

Try  $z^2$  (Example) :  $\underline{\Phi}(z) = z^2 = (x+iy)(x+iy)$

$$= x^2 + 2ixy - y^2$$

$$= \underbrace{x^2 - y^2}_{\varphi} + i \underbrace{2xy}_{\psi}$$

1.  $\varphi$  &  $\psi$  satisfy  $\nabla^2 \varphi = 0$  and  $\nabla^2 \psi = 0$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2-2=0 \checkmark$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \checkmark$$

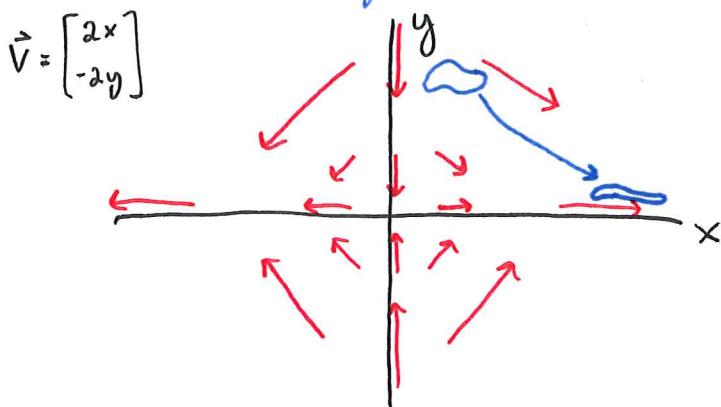
2. Compute  $\vec{V}$

$$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

3.  $\vec{V}$  is incomp. & irrot.

$$\nabla \cdot \vec{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} 2x \\ -2y \end{bmatrix} = 0 \quad \checkmark$$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2y & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} \quad \checkmark$$



$$\begin{cases} \dot{x} = v_1(x,y) \\ \dot{y} = v_2(x,y) \end{cases} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

ODE for particle in induced velocity field.

## 2D Potential Flow

$$\underbrace{\Phi(z)}_{\text{complex potential}} = \underbrace{\varphi(x, y)}_{\text{potential function}} + i \underbrace{\psi(x, y)}_{\text{stream function}}, \quad z = x + iy$$

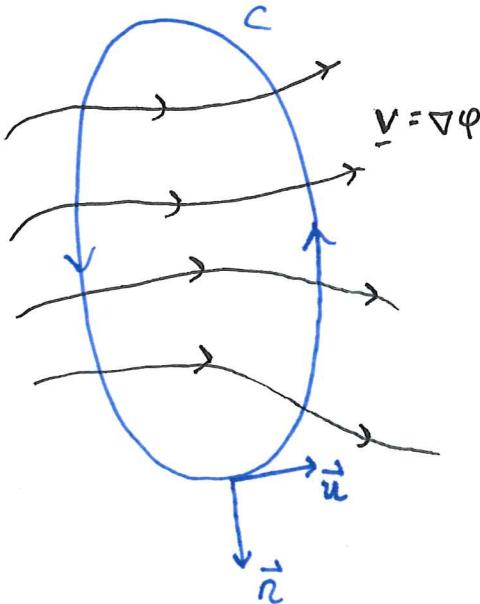
If  $\Phi(z)$  is a nice analytic complex function,

then both  $\varphi$  and  $\psi$  satisfy Laplace's Eq<sup>n</sup>:

$$\nabla^2 \varphi = 0$$

$$\nabla^2 \psi = 0$$

We have  $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \nabla \varphi$  (i.e. the grad of  $\varphi$  defines a vector field).



$$\oint_C \underline{v} \cdot \vec{u} ds = \iint_{\text{inside } C} \nabla \times \underline{v} dA = 0$$

Zero circulation around  $C$ .

$$\oint_C \underline{v} \cdot \vec{n} ds = \iint_{\text{inside } C} \nabla \cdot \underline{v} dA = 0$$

Zero flux across  $C$ .

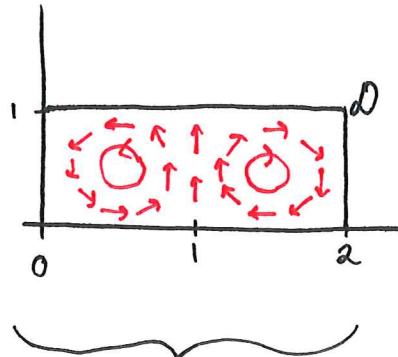
true for  $\vec{v} = \nabla \varphi$  and  $\nabla^2 \varphi = 0$ .

## Example

$$\Psi(x, y) = \sin(\pi x) \sin(\pi y) \quad \text{on } D = [0, 2] \times [0, 1]$$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial \Psi / \partial y \\ -\partial \Psi / \partial x \end{bmatrix}$$

$$= \begin{bmatrix} \pi \sin(\pi x) \cos(\pi y) \\ -\pi \cos(\pi x) \sin(\pi y) \end{bmatrix}$$



Double Gyre Flow

More on this next time!