

## Overview of Topics

### ① $\nabla$ , $\nabla \cdot$ , and $\nabla \times$

- $\nabla f = \text{grad } f$
- $\nabla \cdot \vec{f} = \text{div } \vec{f}$
- $\nabla \times \vec{f} = \text{curl } \vec{f}$

# Del, Nabla, and vectors of partial derivatives

Called 'Del'  
or 'Nabla' →  $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$

... but  
"Nabla tha funky homosapien"  
doesn't have the same ring  
to it... OR

$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

$\nabla$  is a linear operator, meaning that

$$\nabla(f+g) = \nabla f + \nabla g$$

$$\text{and } \nabla(\alpha f) = \alpha \nabla f \quad (\text{for all scalars } \alpha)$$

# Gradient

When used as a linear operator on  
a scalar function  $f$ ,  $\nabla$   
is called "the gradient":

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

(now  $\nabla f(x)$  is a vector,  
defining a vector field!)

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Example: ~~Find the gradient field~~

# Example of Gradient: Gravitational Force Field!

Newton's Law of Gravity:

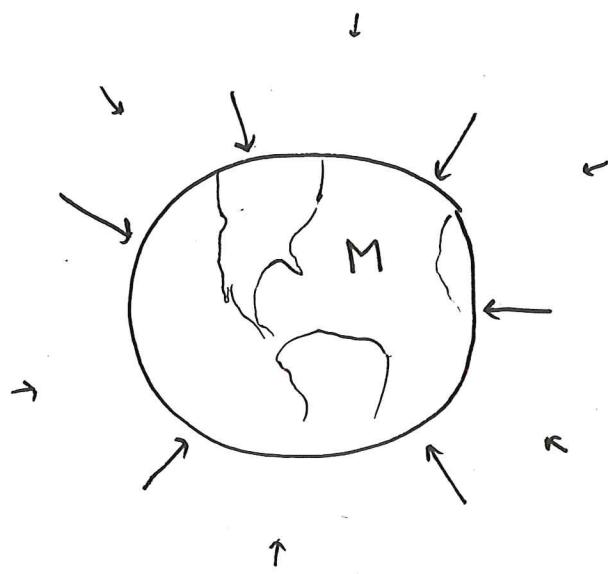
$$\vec{F} = -\frac{mM G}{r^3} \hat{r} = -\nabla V$$

object mass    earth mass    Grav. constant

$r = \sqrt{x^2 + y^2 + z^2}$

Gravitational Potential

$$V = -\frac{mM G}{r}$$



Note, that for bead on hoop  
we were linearizing the gravitational potential about the earth Radius  $R_0$ ,  
since experiments were conducted on earth surface in earth gravity  $g$ .  
Try Taylor expanding  $\frac{1}{R+h} \dots$

In  $x, y, z$  components:

$$\frac{\partial V}{\partial x} = \frac{mM G x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial V}{\partial y} = \frac{mM G y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial V}{\partial z} = \frac{mM G z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F = -\nabla V =$$

$$\begin{bmatrix} -\frac{mM G}{r^3} x \\ -\frac{mM G}{r^3} y \\ -\frac{mM G}{r^3} z \end{bmatrix}$$

# Divergence

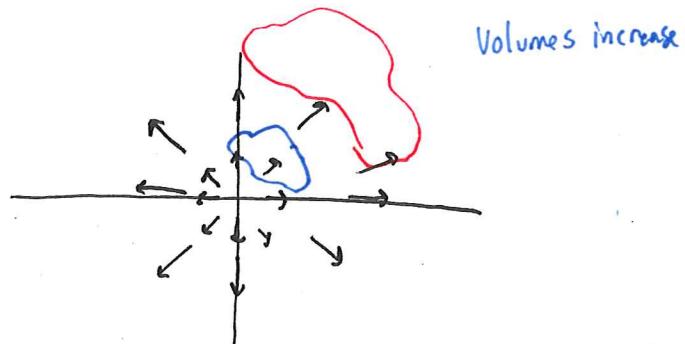
Since  $\nabla$  is a vector, we may take inner product with other vectors  $\vec{f}$ :

$\nabla \cdot \vec{f}$  is the "Divergence of  $\vec{f}$ ".

$$\text{Ex 1: } \vec{f}(x,y) = x\vec{i} + y\vec{j} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\nabla \cdot \vec{f} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 1+1=2.$$

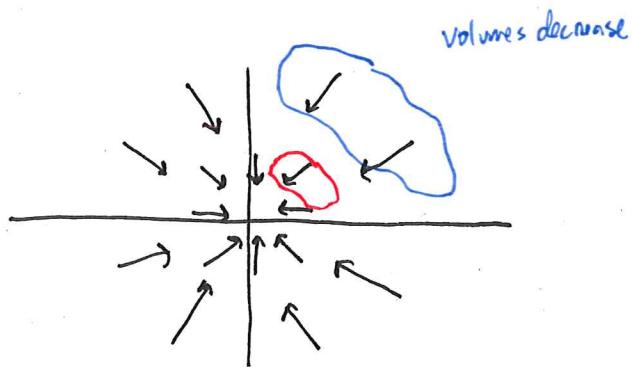
"The flow is diverging"



$$\text{Ex 2: } \vec{f}(x,y) = \cancel{x\vec{i}} \cancel{y\vec{j}} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

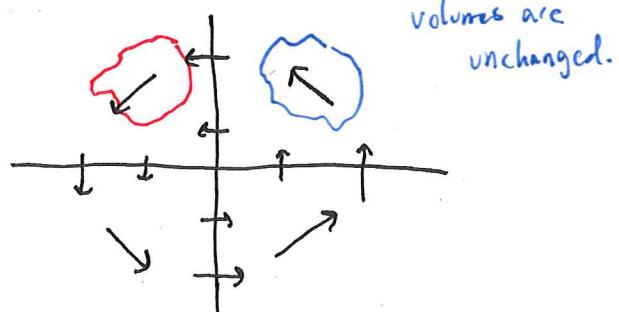
$$\nabla \cdot \vec{f} = -2$$

"The flow is converging"



$$\text{Ex 3: } \vec{f}(x,y) = -y\vec{i} + x\vec{j} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\nabla \cdot \vec{f} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = 0$$



"The flow is incompressible"

(i.e. zero divergence).

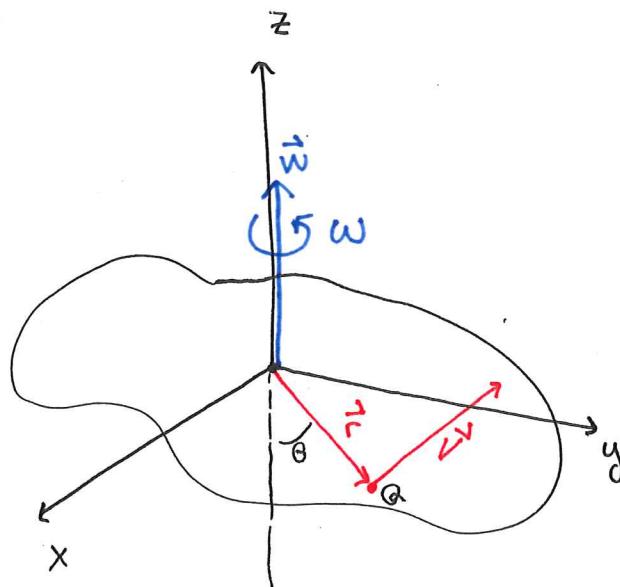
Curl of  $\vec{f}$  is  $\nabla \times \vec{f}$

$$\text{Let } \vec{f}(x, y, z) = \begin{bmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{bmatrix}$$

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}$$

$$\text{Ex} \quad \vec{f} = xy \vec{i} - \sin(z) \vec{j} + \vec{k}$$

$$\nabla \times \vec{f} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{bmatrix} = \frac{\partial}{\partial z} \sin(z) \vec{i} - \frac{\partial}{\partial y} xy \vec{k} \\ = \cos(z) \vec{i} - x \vec{k}$$



Velocity of point Q

$$\text{is } \vec{v} = \vec{w} \times \vec{r}$$

where  $\vec{w}$  is the angular velocity vector

$$\text{with magnitude } \|\vec{w}\| = \omega$$

$$\text{If } \vec{w} = \omega \vec{k} \text{ and } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} = \vec{w} \times \vec{r} = -\omega y \vec{i} + \omega x \vec{j}$$

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \vec{k} = 2\vec{w}.$$

For solid body rotation, curl of velocity vector is a vector field w/ constant magnitude everywhere. The VF points in axis of rotation with magnitude twice the angular speed.

## Important Facts

Curl of gradient = 0

$$\nabla \times (\nabla f) = 0 \quad \text{for all } f \text{ (smooth)}$$

div of curl = 0

$$\nabla \cdot (\nabla \times \vec{f}) = 0 \quad \text{for all } \vec{f} \text{ (smooth)}$$