

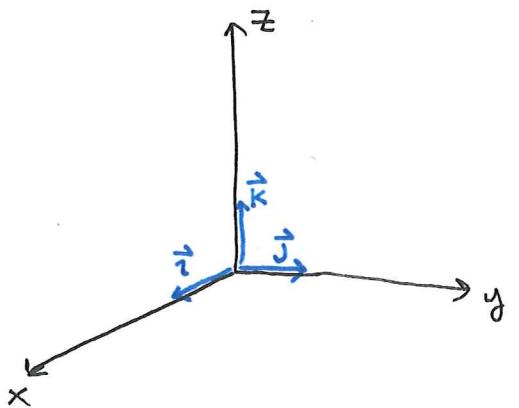
Overview of Topics

① Linear Algebra in 2D and 3D

- Inner product of two vectors
- Norm of a vector
- Cross product of two vectors.

Vectors

Consider a three dimensional space with coordinate directions x , y , and z .



The unit vectors \vec{i} , \vec{j} , and \vec{k}

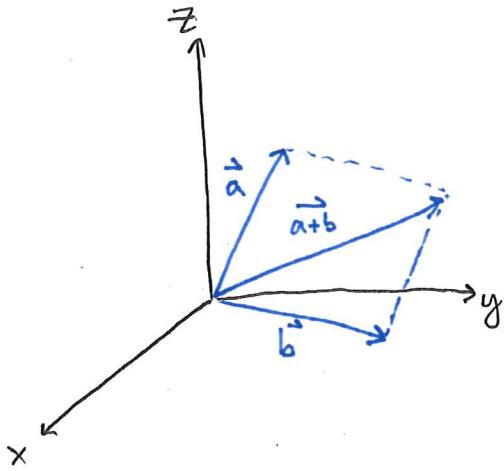
are unit vectors along the three coordinate axes.

\vec{i} , \vec{j} , and \vec{k} form an orthogonal basis for our 3D vector space.

(more on vector spaces soon...)

This space is also ^{sometimes} called a Euclidean space because it is flat (i.e. all vector directions are straight)

and it has the same geometry no matter where you put the origin...

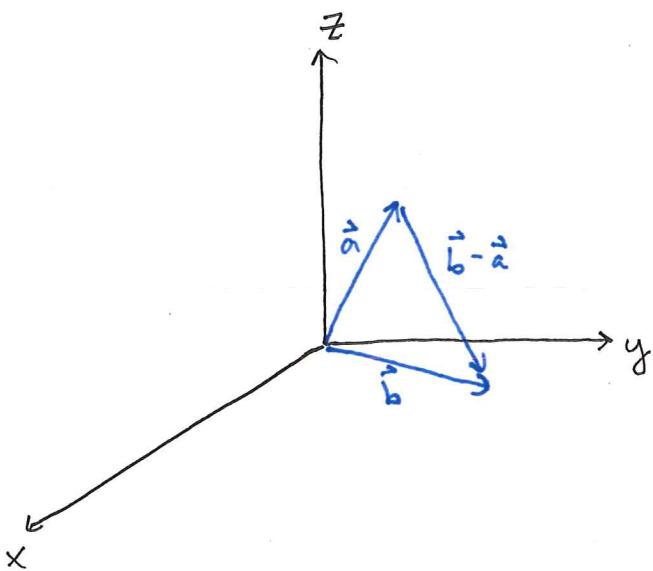


Vector addition

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\vec{a} + \vec{b} = (a_1 + b_1) \vec{i} + (a_2 + b_2) \vec{j} + (a_3 + b_3) \vec{k}$$



Vector subtraction

Note, we often denote vectors as 3×1 column matrices:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Inner Products

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (\text{scalar})$$

$\langle \vec{a}, \vec{b} \rangle$ is also common notation (Dirac & quantum mech.)

$$\vec{a}^T \vec{b} = [a_1 \ a_2 \ a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ also works...}$$

Norm:

From the Pythagorean Theorem, the length (norm)

of $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ is $\sqrt{a_1^2 + a_2^2 + a_3^2}$

$$\xrightarrow{\substack{\text{norm} \\ \text{length}}} \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = (\vec{a} \cdot \vec{a})^{1/2}$$

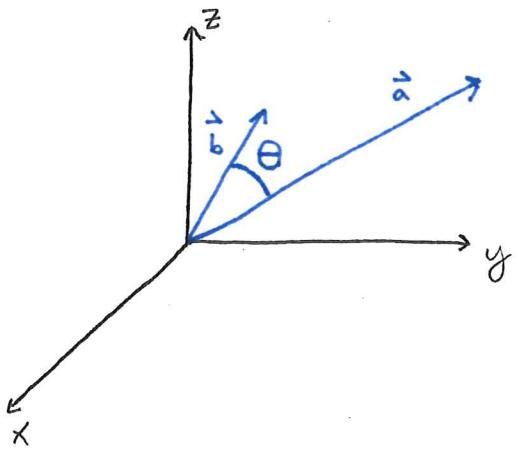
Normalize:

$$\frac{\vec{a}}{\|\vec{a}\|}$$

is a unit vector (i.e. unit length)

in the direction of \vec{a} .

(in other words, we have normalized \vec{a} .)



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

(from Law of Cosines in Trigonometry)

Lets say \vec{a} and \vec{b} are unit vectors

$$\text{so } \|\vec{a}\| = \|\vec{b}\| = 1$$

Then $\vec{a} \cdot \vec{b} = \cos(\theta)$

If parallel, $\vec{a} \cdot \vec{b} = 1$

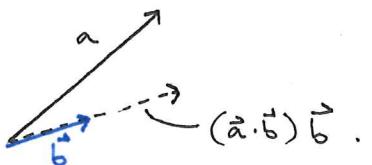
If perpendicular $\vec{a} \cdot \vec{b} = 0$.

We can also solve for θ : $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right)$.

If \vec{b} is a unit vector,

then $(\vec{a} \cdot \vec{b}) \vec{b}$ is the
projection of \vec{a} onto the direction \vec{b}

i.e. how much of \vec{a}
points in the \vec{b} direction...



For arbitrary \vec{a}, \vec{b} the orthogonal projection of

\vec{a} onto \vec{b} is

$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}$$

Cauchy-Schwarz Inequality

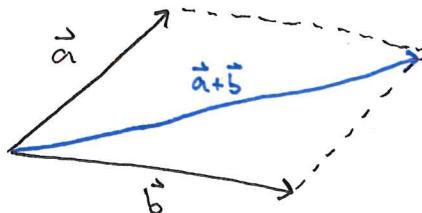
Very useful!

for any vectors \vec{a}, \vec{b} ,

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

Triangle Inequality

even more useful!



$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

only equal if \vec{a}, \vec{b} are parallel!

Proof:
$$\begin{aligned} \|\vec{a} + \vec{b}\|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} \\ &= \|\vec{a}\|^2 + 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 \end{aligned}$$

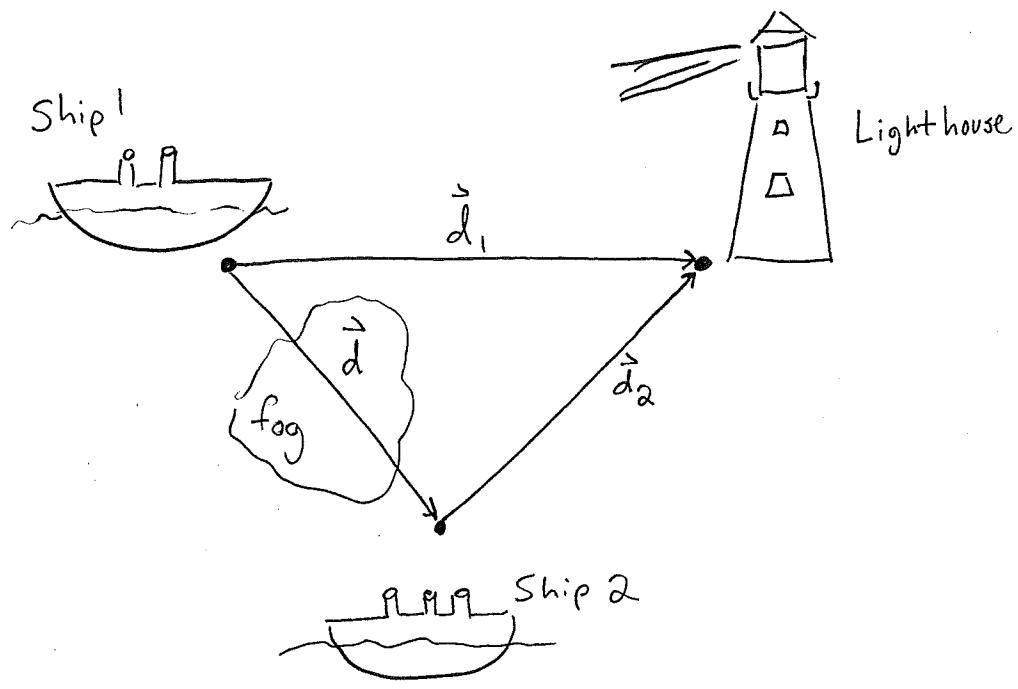
by Cauchy Schwarz:

$$\|\vec{a}\|^2 + 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 \leq \|\vec{a}\|^2 + 2\|\vec{a}\| \|\vec{b}\| + \|\vec{b}\|^2 = (\|\vec{a}\| + \|\vec{b}\|)^2$$

Thus, $\|\vec{a} + \vec{b}\|^2 \leq (\|\vec{a}\| + \|\vec{b}\|)^2$

which is still true if we take square roots.





$$\vec{d} = \vec{d}_1 - \vec{d}_2$$

Cross - Product

For two vectors $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

and $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$

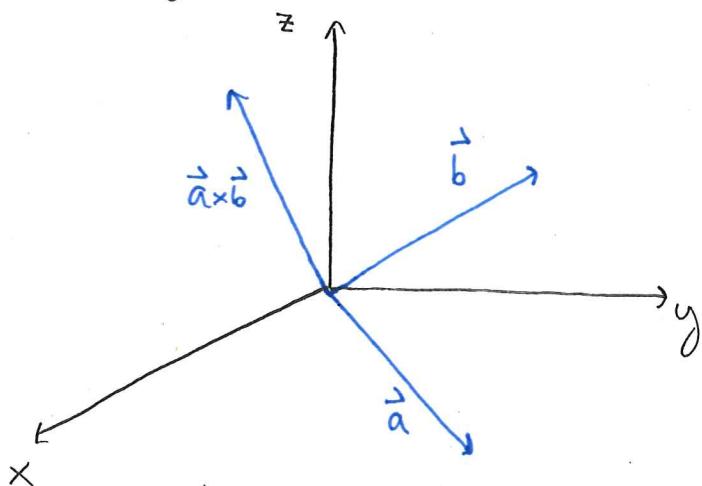
The cross-product $\vec{a} \times \vec{b}$ is

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \vec{i} - \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \vec{j} + \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \vec{k}$$

The result is a vector.

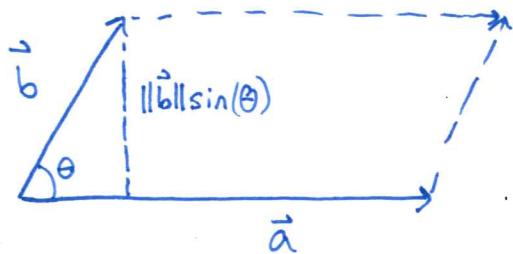
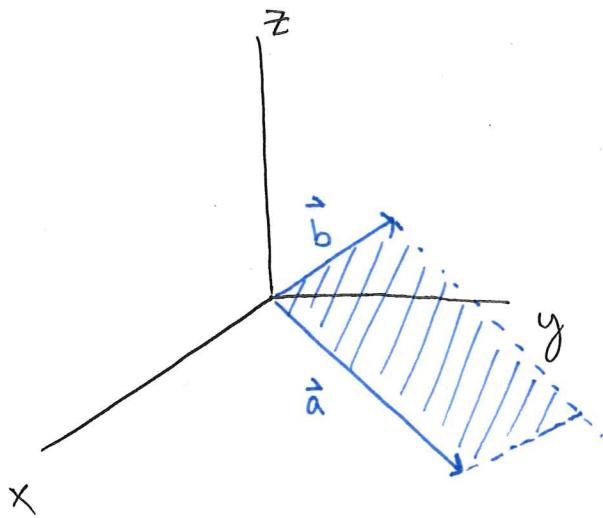
Moreover, the resulting vector is perpendicular to both \vec{a} and \vec{b} and satisfies the right hand rule.



$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{i} - (a_1 b_3 - a_3 b_1) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \text{algebra} = 0$$

Verify by hand... homework!



$$\text{Area} = \|\vec{a}\| \|\vec{b}\| \sin(\theta) = \|\vec{a} \times \vec{b}\|$$

$$\begin{aligned}
 \|\vec{a} \times \vec{b}\|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_1 b_3 - b_1 a_3)^2 + (a_1 b_2 + a_2 b_1)^2 \\
 &= \dots = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
 &= \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2 \theta \\
 &= \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta \quad \blacksquare
 \end{aligned}$$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{i} - (a_1 b_3 - a_3 b_1) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_1 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrix form
of " $\vec{a} \times$ "

Skew-symmetric matrix...