L17 : November 3, 2014 ME 564, Fall 2014

# Overview of Topics

① Numerical Solutions to ODEs

Forward & Backward Euler:

(a) Numerical Example

(b) Stability

(c) Error

$\dot{x} = f(x), \quad x(0) = x_0$

$x$ - may be vector of states
$f$ - may be nonlinear function.

$\dot{x} = Ax, \; x(0) = x_0$ is much simpler for matrix $A$.
system of first-order linear differential Eq's.

$x(t) = e^{At} x_0$ ... different class.

We are interested in numerically solving this, by starting with $x_0$ and iterating to get $x_0 \to x_1 \to x_2 \to \cdots \to x_N$. (trajectory)

Forward Euler:

$$\frac{x_{k+1} - x_k}{\Delta t} \approx \dot{x} = f(x_k) \implies \boxed{x_{k+1} = x_k + \Delta t\, f(x_k)}$$

$$\text{If } \dot{x} = Ax \implies \boxed{x_{k+1} = (\underbrace{I}_{\text{identity matrix.}} + \Delta t A) x_k}$$

(not very stable)

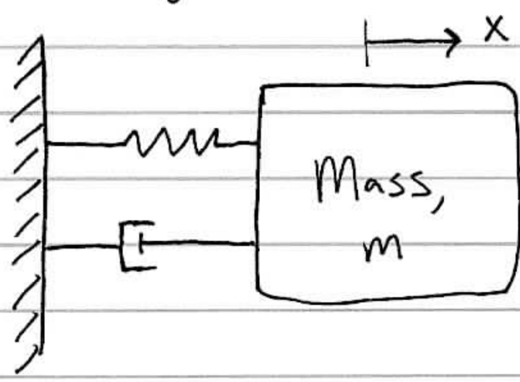
Backward (Implicit) Euler:

$$\frac{x_{k+1} - x_k}{\Delta t} \approx f(x_{k+1}) \implies x_{k+1} = x_k + \Delta t\, f(x_{k+1})$$

$$\text{if } \dot{x} = Ax \implies x_{k+1} = x_k + A \Delta t\, x_{k+1}$$

$$\implies \boxed{x_{k+1} = (I - A\Delta t)^{-1} x_k}$$ better stability.

# Spring-Mass-Damper




$$m\ddot{x} = -kx - c\dot{x}$$

$$m\ddot{x} + kx + c\dot{x} = 0$$

$$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = 0$$

If $\omega_0 = \sqrt{\frac{k}{m}}$ natural frequency

$\zeta = \frac{c}{2\sqrt{km}}$ damping ratio.

$$\boxed{\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0}$$

Second order <u>linear</u> differential equation.

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2\zeta\omega_0 v - \omega_0^2 x \end{aligned}\right\} \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}}\begin{bmatrix} x \\ v \end{bmatrix}$$

$\omega_0$ & $\zeta$ determine eigenvalues of $A$, hence, the behavior of the system.

Cases: ① Under-damped $\zeta < 1$

system oscillates w/ freq $\omega_d = \omega_0 \sqrt{1-\zeta^2}$

② Over-damped $\zeta > 1$

③ Critically Damped $\zeta = 1$

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Lets code up forward Euler

$$x_{k+1} = (I + A\Delta t)\, x_k$$

... try $dt = .01$ $T = 10$

... compare w/ RK4

... try $dt = 0.1$, $d = 0.5$, $d = 1$, $d = 2$.

What went wrong?...

Look at eig$(I + A\Delta t)$.

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Next time:

- error analysis
- global stability properties
- why is ODE45 so good?

```
clear all

w = 2*pi;
d = 1.75;  % will break for d=20

A = [0 1; -w^2 -2*d*w];

dt = .1;  % time step
T = 10;    % amount of time to integrate

x0 = [2; 0];  % initial condition

% iterate forward euler
xF(:,1) = x0;
tF(1) = 0;
for i=1:T/dt
    tF(i+1) = i*dt;
    xF(:,i+1) = (eye(2) + A*dt)*xF(:,i);
end
plot(tF,xF(1,:),'k')
hold on

% iterate backward euler
xB(:,1) = x0;
tB(1) = 0;
for i=1:T/dt
    tB(i+1) = i*dt;
    xB(:,i+1) = inv(eye(2)-A*dt)*xB(:,i);
end
plot(tB,xB(1,:),'b')

% compute better integral using build-in Matlab code
[t,y] = ode45(@(t,y) A*y, 0:dt:T,x0);
hold on
plot(t,y(:,1),'r')
```

$$\dot{y} = f(y)$$

Forward Euler: $y_{k+1} = y_k + \Delta t\, f(y_k)$

$$y_{k+1} \approx y(t_{k+1})$$
$$= y(t_k + \Delta t)$$

Taylor Expansion:

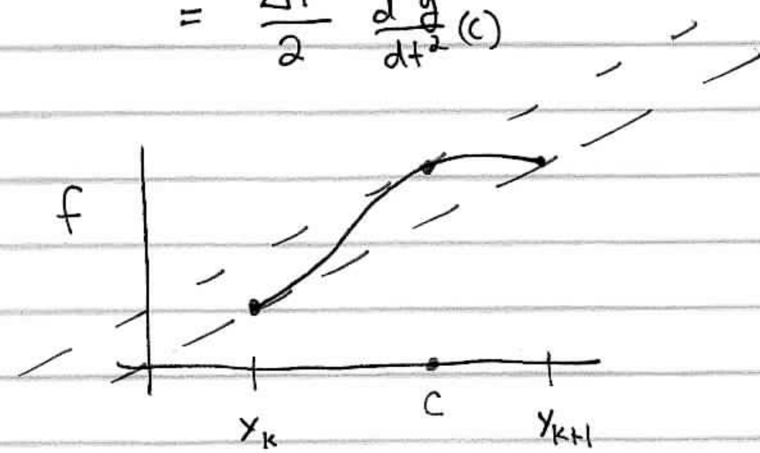
$$\underbrace{y(t_k + \Delta t)}_{\text{exact trajectory}} = y(t_k) + \Delta t \frac{dy}{dt}(t_k) + \frac{\Delta t^2}{2!}\frac{d^2 y}{dt^2}(c)$$

$$c \in [t_k, t_{k+1}]$$

$$= y_k + \Delta t\, f(y_k) + \mathcal{O}(\Delta t^2)$$

Local Error $\quad \varepsilon_{k+1} = y(t_k + \Delta t) - y_{k+1}$

$$= \frac{\Delta t^2}{2}\frac{d^2 y}{dt^2}(c)$$




Global Error $\quad E_k = \sum_{j=0}^{k} \varepsilon_k \approx \frac{b-a}{2}\Delta t \frac{d^2 f}{dt^2}(c)$

$$\sim \mathcal{O}(\Delta t)$$

# Stability

**Ex 1** $\quad \dot{y} = \lambda y \qquad y(0) = y_0$

Forward Euler: $\quad y_{k+1} = (1+\lambda\Delta t) y_k$

Backward Euler: $\quad y_{k+1} = \underbrace{(1-\lambda\Delta t)^{-1}}_{eig(\text{" "})} y_k$

**FE** : $\quad y_0 \to y_1 \to y_2 \to \dots \to y_N$

$$y_N = (1+\lambda\Delta t)^N y_0$$

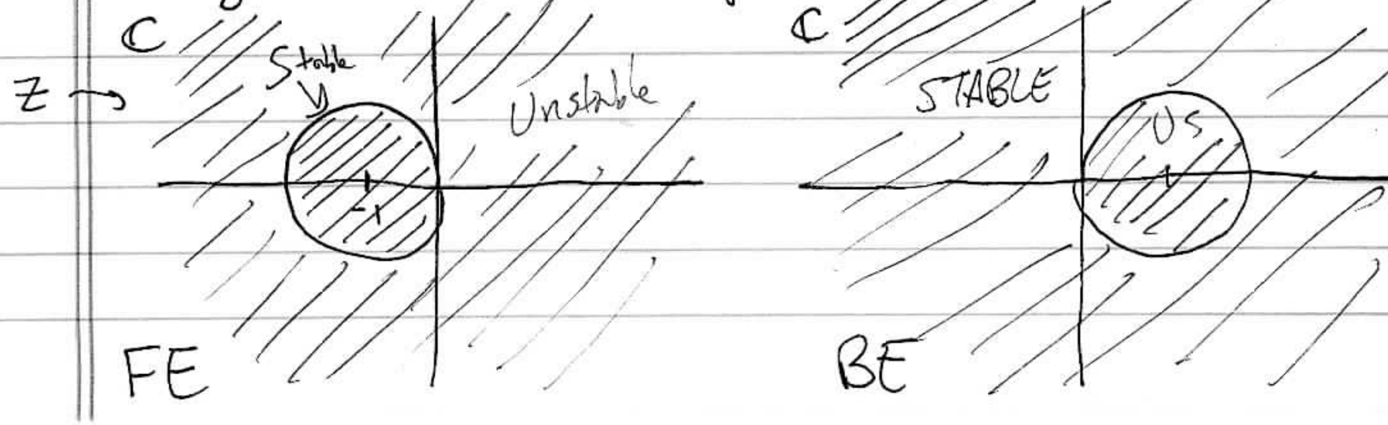
$$\boxed{\text{Unstable when } |1+\lambda\Delta t| > 1}$$

**BE** : $\quad y_0 \to y_1 \to \dots$

$$y_N = \left(\frac{1}{1-\lambda\Delta t}\right)^N y_0$$

$$\boxed{\text{Unstable when } \left|\frac{1}{1-\lambda\Delta t}\right| > 1}$$

Say $z = \lambda\Delta t$ is Complex #.



Ex 2 $\quad \dot{y} = Ay \qquad y(0) = y_0$

FE: $\quad y_{k+1} = (I + \Delta t A)\, y_k$

BE: $\quad y_{k+1} = (I - \Delta t A)^{-1} y_k$

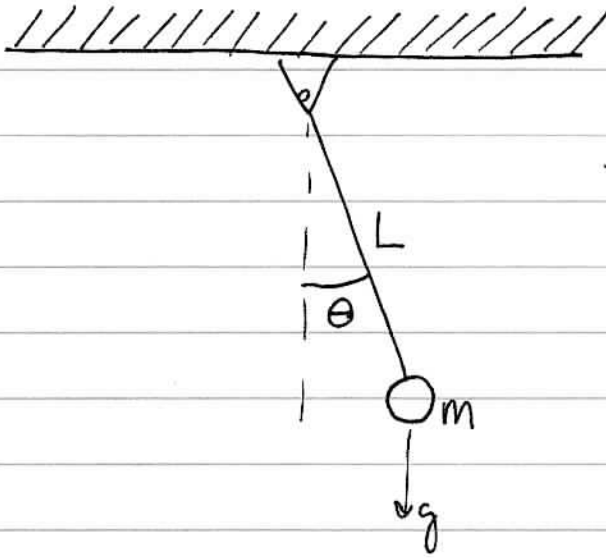
FE: $\quad y_N = (I + \Delta t A)^N y_0$

$$\boxed{\text{Stable when } \left| \operatorname{eigs}(I + \Delta t A) \right| < 1}$$

BE: $\quad y_N = (I - \Delta t A)^{-N} y_0$

$$\boxed{\text{stable when } \left| \operatorname{eigs}\left((I - \Delta t A)^{-1}\right) \right| < 1}$$

Single-Pendulum




$$T = \frac{1}{2} m L \dot{\theta}^2$$

$$V = Lmg\left(1 - \cos(\theta)\right)$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \implies \boxed{\ddot{\theta} = \frac{g}{L}\sin(\theta)}$$

$$\ddot{x} = \frac{g}{L}\sin(x) \quad \ldots \quad \text{say } g/L = 1$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = \sin(x) \end{cases} \implies \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \end{bmatrix} \leftarrow \underline{f}(\underline{x}).$$

$$\underline{\dot{x}} = \underline{f}(\underline{x}).$$

Simulate in PPLANE8.m

Dynamical Systems
Mechanical / Aerospace
Electrical Engineering
Physics
Biology
...

```
function dy = pend(t,y,g,L,d)
dy(1,1) = y(2);
dy(2,1) = (g/L)*sin(y(1))-d*y(2);
```

```
%dy = pend(t,y,g,L,d)

clear all

t = 0:.1:50;
y0 = [pi/4; 0];

g = -10;
L = 10;
d = 0.1;
[t,y] = ode45(@(t,y)pend(t,y,g,L,d),t,y0);

figure
plot(t,y(:,1));

figure
plot(y(:,1),y(:,2));

figure
plot3(t,y(:,1),y(:,2))
```