$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{N-1} f(x_k)\Delta x \quad \text{(Left-sided)}$$

not very precise

$$\int_{x_0}^{x_0+\Delta x} f(x)\,dx = \int_{x_0}^{x+\Delta x} \left[f(x_0) + \Delta x \frac{df}{dx}(x_0) + \frac{\Delta x^2}{2!}\frac{d^2f}{dx^2}(x_0) + \dots \right] dx$$

$$= \Delta x\, f(x_0) + \underbrace{\Delta x^2 \frac{df}{dx}(x_0) + \frac{\Delta x^3}{2!}\frac{d^2f}{dx^2}(x_0) + \dots}_{\text{error}}$$

More carefully

$$\int_{x_0}^{x_0+\Delta x} \left[f(x_0) + (x-x_0)\frac{df}{dx}(x_0) + \frac{(x-x_0)^2}{2!}\frac{d^2f}{dx^2}(x_0) + \dots \right] dx$$

$$= \left[f(x_0)\,x + \frac{(x-x_0)^2}{2}\frac{df}{dx}(x_0) + \frac{(x-x_0)^3}{3!}\frac{d^2f}{dx^3}(x_0) + \dots \right]_{x_0}^{x_0+\Delta x}$$

$$= f(x_0)\Delta x + \underbrace{\frac{\Delta x^2}{2!}\frac{df}{dx}(x_0)}_{\mathcal{O}(\Delta x^2)} + \dots$$

$\mathcal{O}(\Delta x^2)$ at each step...

but we take $\frac{b-a}{\Delta x}$ steps!

so total error $= \mathcal{O}(\Delta x)$.

$$\sum_{k=0}^{N-1} \frac{1}{2}\left(f(x_k) + f(x_{k+1})\right)\Delta x$$

evaluates f a lot of times... what if f is expensive?

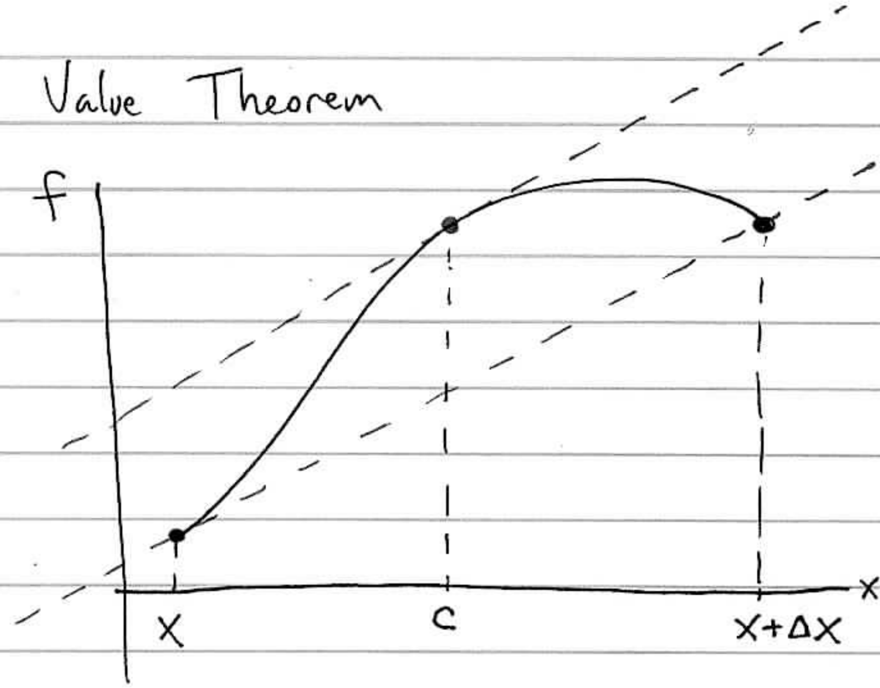
$$= \frac{\Delta x}{2}\left[f(x_0) + f(x_N) + 2\sum_{k=1}^{N-1} f(x_k)\right]$$

Not going to talk about Recursive Improvement, pg 72.

Example: f with very large derivatives.

- could use smaller Δx
- could use higher order scheme.

Mean Value Theorem



$$\frac{df}{dx}(c) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

So $$f(x+\Delta x) = f(x) + \Delta x \frac{df}{dx}(c) \, . \quad (\text{no h.o.t.})$$

Can do for higher derivative terms:

$$f(x+\Delta x) = f(x) + \Delta x \frac{df}{dx}(x) + \frac{\Delta x^2}{2!} \frac{d^2 f}{dx^2}(x) + \underset{\cdots}{\text{h.o.t.}}$$

$$= f(x) + \Delta x \frac{df}{dx}(x) + \frac{\Delta x^2}{2!} \frac{d^2 f}{dx^2}(c) \, .$$

... so on.