L15: Oct. 29, 2014 ME 564, Fall 2014

Overview of Topics

1. Numerical Differentiation on **DATA**

2. Machine precision

3. Intro to numerical integration
   (Rectangles & Riemann integrals)

Last Time: Finite difference apx. to $f'(x)$

Forward difference $\quad \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \quad \mathcal{O}(\Delta x)$ error

Backward difference $\quad \dfrac{f(x) - f(x-\Delta x)}{\Delta x} \quad \mathcal{O}(\Delta x)$ error

Central difference $\quad \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \quad \mathcal{O}(\Delta x^2)$ error

error analysis involves Taylor expansions...

can get <u>higher</u> accuracy schemes by using more points: i.e. $f(x+2\Delta x)$, $f(x-2\Delta x)$, etc.

Second derivative? $f''(t)$

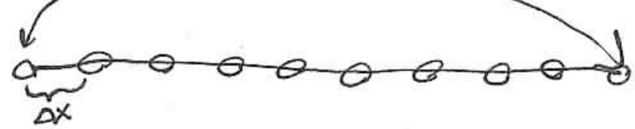
$$f(t+\Delta t) + f(t-\Delta t) = 2f(t) + \Delta t^2 \frac{d^2 f(t)}{dt^2} + \frac{\Delta t^4}{4!}\left(\frac{d^4 f(t)}{dt^4}\right) + \mathcal{O}(\Delta t^6).$$

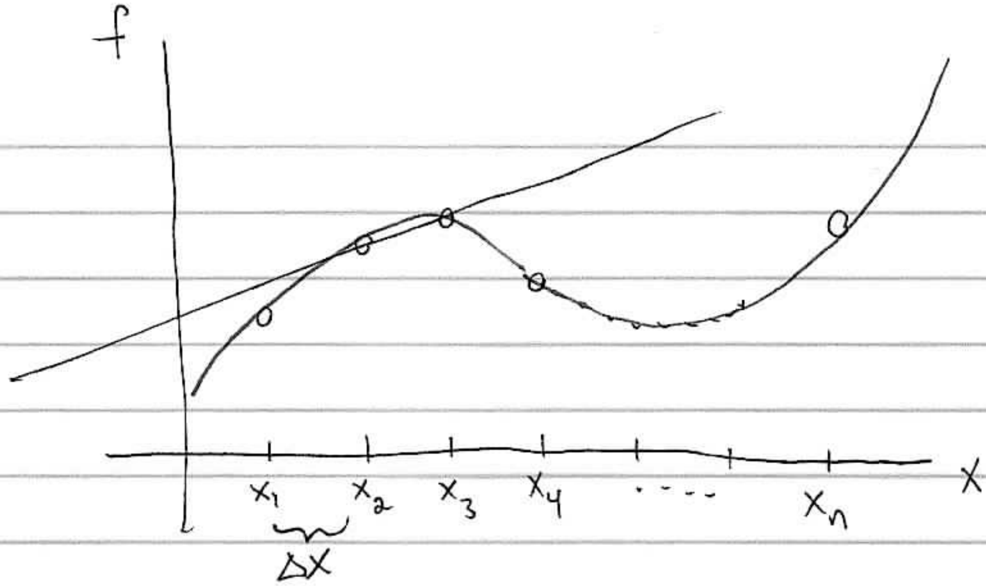
$$\frac{d^2 f}{dt^2}(t) = \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} + \mathcal{O}(\Delta t^2)$$

(looks a lot like what we would get if we "finite differenced" starting with $f'(x)$, $f'(x+\Delta x)$, $f'(x-\Delta x)$...)

Central difference is generally better (when possible!):

- not possible when computing $f'(t)$ in real-time
- not possible when computing $f'(x)$ at boundaries of x data





$x_n - x_{n-1} = \Delta x$

x_1	f_1
x_2	f_2
x_3	f_3
$\vdots$	$\vdots$
x_n	f_n

Use central difference
on "interior points"
$x_2, x_3, \ldots x_{n-1}$

Use forward for x_1 (are Δx^2 schemes)
backward for x_n (Δx^2 schemes)

```
clear all

% numerically differentiate sin(x) on a discrete grid.
% compare with exact derivative (cos(x))

x = .1:.1:3;
f = sin(x);

plot(x,f,'k')
hold on
plot(x,f,'rx','LineWidth',2)

dx = x(2)-x(1);

n = length(f);

dfdx = zeros(n,1);

dfdx(1) = (f(2)-f(1))/(x(2)-x(1));  % forward diff at f(x_1)
for i=2:n-1
    dfdx(i)  = (f(i+1)-f(i-1))/(x(i+1)-x(i-1)); % central in between
end
dfdx(n) = (f(n)-f(n-1))/(x(n)-x(n-1));  % backward diff at f(x_n)

figure
plot(x,cos(x),'k')
hold on
plot(x,dfdx,'r')
```

as $\Delta t \to 0$ does error

become arbitrarily small?

No! Answer: numerical truncation error

roundoff error ($e_r \sim 10^{-16}$ for double precision)

$$A = A + e_r/2$$

$$\frac{df}{dt} = \frac{f(\cancel{*}t+\Delta t) - f(t-\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

$$\frac{df}{dt} = \frac{f(t+\Delta t) - f(t-\Delta t) + 2e_r}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

$$|\text{Error}| \le \underbrace{\frac{\cancel{2}e_r}{\cancel{2}\Delta t} + \Delta t^2 \frac{M}{6}}_{E_{max}}$$

$M = \max \frac{d^3 f(c)}{dx^3}$, c in $x-\Delta t$ to $x+\Delta t$*

$$\frac{\partial E_{max}}{\partial \Delta t} = -\frac{e_r}{\Delta t^2} + \frac{\Delta t M}{3} = 0$$

$$\Rightarrow \Delta t = \left(\frac{3e_r}{M}\right)^{1/3}$$

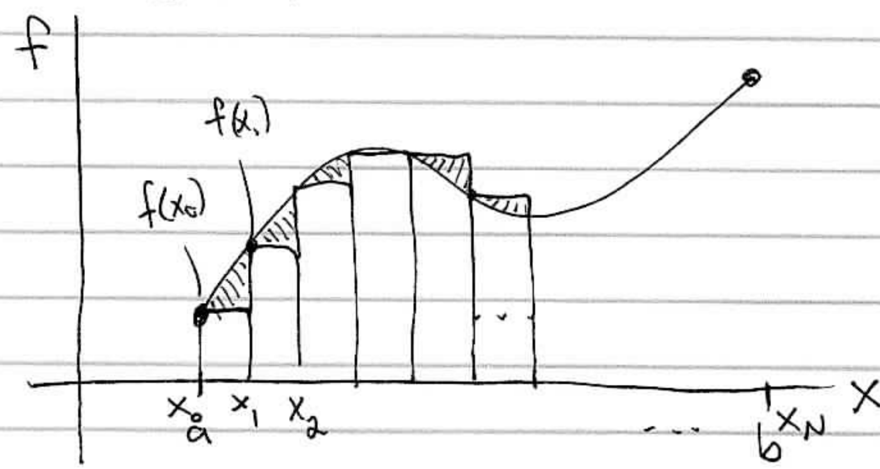
$$\approx \underline{\underline{10^{-5}}}$$

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} \left(f\left(a + \frac{b-a}{N} k\right)\right)\left(\frac{b-a}{N}\right)$$

$$= \lim_{\Delta x\to 0} \sum_{k=1}^{N} f(a + k\Delta x)\,\Delta x$$

$$\boxed{= \lim_{\Delta x\to 0} \sum_{k=1}^{N} f(x_k)\,\Delta x}$$ Right-sided rectangle

$$= \lim_{\Delta x\to 0} \sum_{k=0}^{N-1} f(x_k)\,\Delta x$$ Left-sided



N is # of rectangles

$\Delta x = (b-a)/N$ - width