Overview of Topics:

0. Brief midterm review!
1. Numerical Differentiation

   (a) Finite difference for $f'(x)$

   (b) Finite difference for $f''(x)$

   (c) Numerical Roundoff in computations

$$\dot{x} = -2x + e^{t} \qquad x_0 = 5.$$

$$x(t) = e^{-2t}x_0 + \int_0^t e^{-2(t-\tau)}e^{\tau}\,d\tau$$

$$= 5e^{-2t} + e^{-2t}\int_0^t e^{2\tau}e^{\tau}\,d\tau$$

$$= 5e^{-2t} + e^{-2t}\int_0^t e^{3\tau}\,d\tau$$

$$= 5e^{-2t} + e^{-2t}\left[\frac{1}{3}e^{3\tau}\right]_0^t$$

$$= 5e^{-2t} + e^{-2t}\left[\frac{1}{3}e^{3t} - \frac{1}{3}\right]$$

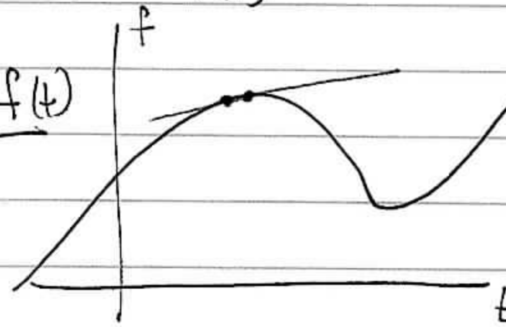
$$\boxed{x(t) = \left[5 - \tfrac{1}{3}\right]e^{-2t} + \frac{1}{3}e^{t}}$$

Example w/ convolution.

Numerical Differentiation

- given function (or data) $f(t)$

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$



$(*)$ $$f(t+\Delta t) = f(t) + \Delta t \frac{df}{dt}(t) + \frac{\Delta t^2}{2!}\frac{d^2f}{dt^2}(t) + \frac{\Delta t^3}{3!}\frac{d^3f}{dt^3}(t) + \ldots \text{h.o.t.} + \mathcal{O}(\Delta t^4)$$

$(**)$ $$f(t-\Delta t) = f(t) - \Delta t \frac{df}{dt}(t) + \frac{\Delta t^2}{2!}\frac{d^2f}{dt^2}(t) - \frac{\Delta t^3}{3!}\frac{d^3f}{dt^3}(t) + \ldots + \text{h.o.t.}$$

Forward Difference $$\frac{f(t+\Delta t) - f(t)}{\Delta t} = \underline{\underline{\frac{df}{dt}(t)}} + \frac{\Delta t}{2!}\frac{d^2f}{dt^2}(t) + \frac{\Delta t^2}{3!}\frac{d^3f}{dt^3} + \text{h.o.t.}$$

Error on the order of Δt

Backward Difference $$\frac{f(t) - f(t-\Delta t)}{\Delta t} = \frac{df}{dt}(t) - \frac{\Delta t}{2!}\frac{d^2f}{dt^2}(t) + \frac{\Delta t^2}{3!}\frac{d^3f}{dt^3}(t) + \mathcal{O}(\Delta t^3)$$

Error on the order of Δt

Error $\sim \mathcal{O}(\Delta t)$

Central Difference $\underline{(*) - (**)}$ :: $f(t+\Delta t) - f(t-\Delta t)$

$$= 2\Delta t \frac{df}{dt}(t) + \frac{2\Delta t^3}{3!}\frac{d^3f}{dt^3}(t) + \mathcal{O}(\Delta t^5)$$

$$\frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} = \frac{df}{dt}(t) + \frac{\Delta t^2}{3!}\frac{d^3f}{dt^3}(t) + \mathcal{O}(\Delta t^4)$$

Error $\sim \mathcal{O}(\Delta t^2)$.

Second derivative

$$\frac{d^2f}{dt^2}(t) = \lim_{\Delta t \to 0} \frac{\frac{df}{dt}(t+\Delta t) - \frac{df}{dt}(t)}{\Delta t}.$$

I can use any of my schemes to get $\frac{df}{dt}(t+\Delta t)$ and $\frac{df}{dt}(t)$.

... could use $\mathcal{O}(\Delta t)$ forward or backward.

... could use $\mathcal{O}(\Delta t^2)$ centred difference

... probably important to be consistent and use same scheme for $\frac{df}{dt}(t+\Delta t)$ and $\frac{df}{dt}(t)$...

- $$f(t+\Delta t) + f(t-\Delta t) = 2f(t) + \Delta t^2 \frac{d^2f}{dt^2}(t) + \mathcal{O}(\Delta t^4)$$

$$\boxed{\frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} = \frac{d^2f}{dt^2}(t) + \mathcal{O}(\Delta t^2).}$$

Try different combinations!

Try and find an $\mathcal{O}(\Delta t^3)$ or $\mathcal{O}(\Delta t^4)$ algorithm for $\frac{df}{dt}(t)$

and for $\frac{d^2f}{dt^2}(t)$...

Last Time: Finite difference apx. to $f'(x)$

Forward difference $\quad \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \quad \mathcal{O}(\Delta x)$ error

Backward difference $\quad \dfrac{f(x) - f(x-\Delta x)}{\Delta x} \quad \mathcal{O}(\Delta x)$ error

Central difference $\quad \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \quad \mathcal{O}(\Delta x^2)$ error

error analysis involves Taylor expansions...

can get <u>higher</u> accuracy schemes by using more points: i.e. $f(x+2\Delta x)$, $f(x-2\Delta x)$, etc.

Second derivative? $f''(t)$

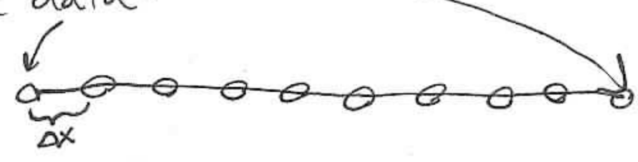
$$f(t+\Delta t) + f(t-\Delta t) = 2f(t) + \Delta t^2 \frac{d^2 f(t)}{dt^2} + \frac{\Delta t^4}{4!}\left(\frac{d^4 f(t)}{dt^4}\right) + \mathcal{O}(\Delta t^6).$$

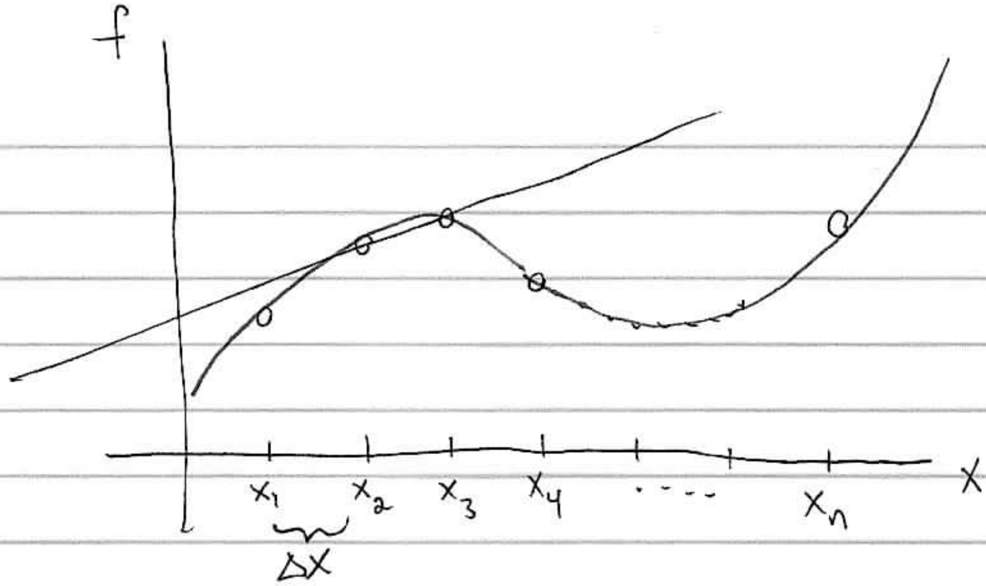
$$\frac{d^2 f}{dt^2}(t) = \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} + \mathcal{O}(\Delta t^2)$$

(looks a lot like what we would get if we "finite differenced" starting with $f'(x)$, $f'(x+\Delta x)$, $f'(x-\Delta x)$...)

Central difference is generally better (when possible!):

- not possible when computing $f'(t)$ in real-time
- not possible when computing $f'(x)$ at boundaries of x data





$x_n - x_{n-1} = \Delta x$

x_1	f_1
x_2	f_2
x_3	f_3
$\vdots$	$\vdots$
x_n	f_n

Use central difference
on "interior points"
$x_2, x_3, \ldots x_{n-1}$

Use forward for x_1 (are Δx^2 schemes)
backward for x_n (Δx^2 schemes)

as $\Delta t \to 0$ does error

become arbitrarily small?

No! Answer: numerical truncation error

roundoff error ($e_r \sim 10^{-16}$ for double precision)

$$A = A + e_r/2$$

$$\frac{df}{dt} = \frac{f(\cancel{*}+\Delta t) - f(t-\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

$$\frac{df}{dt} = \frac{f(t+\Delta t) - f(t-\Delta t) + 2e_r}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

$$|Error| \le \underbrace{\frac{\cancel{2}e_r}{\cancel{2}\Delta t} + \Delta t^2 \frac{M}{6}}_{E_{max}}$$

$M = \max \frac{d^3 f(c)}{dx^3}$, c in $x-\Delta t$ to $x+\Delta t$

$$\frac{\partial E_{max}}{\partial \Delta t} = -\frac{e_r}{\Delta t^2} + \frac{\Delta t M}{3} = 0$$

$$\Rightarrow \Delta t = \left(\frac{3e_r}{M}\right)^{1/3}$$

$$\approx \underline{\underline{10^{-5}}}$$

```matlab
clear all

dt = .2;
t = -2:.1:4;
f = sin(t);

% Exact Derivative
dfdt = cos(t);

% plotting commands
plot(t,f,'k--','LineWidth',1.2)
hold on, grid on
plot(t,dfdt,'k','LineWidth',3)
l1=legend('Function','Exact Derivative');
set(l1,'FontSize',14)
axis([-2 4 -1.5 1.5])

%%
% Forward Difference Approximation
dfdtF = (sin(t+dt)-sin(t))/dt;
% Backward Difference Approximation
dfdtB = (sin(t)-sin(t-dt))/dt;
% Central Difference Approximation
dfdtC = (sin(t+dt)-sin(t-dt))/(2*dt);

plot(t,dfdtF,'b','LineWidth',1.2) % Forward Difference
plot(t,dfdtB,'g','LineWidth',1.2) % Backward Difference
plot(t,dfdtC,'r','LineWidth',1.2) % Central Difference
l2=legend('Function','Exact Derivative','Forward Diff','Backward Diff','Central Diff')
set(l2,'FontSize',14)
```

```
clear all

% numerically differentiate sin(x) on a discrete grid.
% compare with exact derivative (cos(x))

x = .1:.1:3;
f = sin(x);

plot(x,f,'k')
hold on
plot(x,f,'rx','LineWidth',2)

dx = x(2)-x(1);

n = length(f);

dfdx = zeros(n,1);

dfdx(1) = (f(2)-f(1))/(x(2)-x(1));  % forward diff at f(x_1)
for i=2:n-1
    dfdx(i)  = (f(i+1)-f(i-1))/(x(i+1)-x(i-1)); % central in between
end
dfdx(n) = (f(n)-f(n-1))/(x(n)-x(n-1));  % backward diff at f(x_n)

figure
plot(x,cos(x),'k')
hold on
plot(x,dfdx,'r')
```