L13: Oct. 24, 2014 ME 564, Fall 2014

Overview of Topics

① Convolution for systems with forcing

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

(a) impulse response : $u = \delta(t)$

(b) convolution integral for generic $u(t)$

(c) MATLAB commands

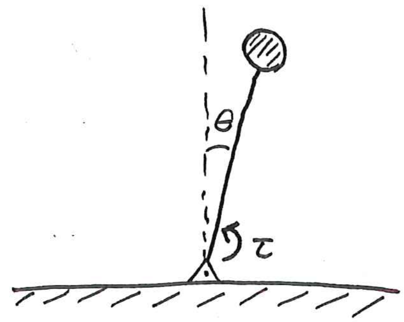
$$\ddot{\theta} = -\sin(\theta) + \tau$$

Example: Stabilizing the inverted pendulum

$$\left.\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\sin(\theta) + \tau\end{aligned}\right\} \quad \frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \underbrace{\begin{bmatrix}\omega \\ -\sin(\theta)\end{bmatrix}}_{f} + \begin{bmatrix}0 \\ \tau\end{bmatrix}$$

at $\theta = \pi$: $\quad \dfrac{Df}{D\underline{x}} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \Longrightarrow \lambda = \pm 1$ (saddle)

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix}}_{\dot{\underline{x}}} = \underbrace{\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}}_{\underline{\underline{A}}}\underbrace{\begin{bmatrix}\theta \\ \omega\end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix}0 \\ 1\end{bmatrix}}_{\underline{\underline{B}}}\underbrace{\tau}_{u}$$



Now, if $\tau = -2\theta - 2\omega$

then $\quad \begin{bmatrix}0 \\ 1\end{bmatrix}\tau = \begin{bmatrix}0 & 0 \\ -2 & -2\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix}$

If I can measure θ, ω then we can feed them back to our control forcing τ!

(for more, see feedback control!)

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix} + \begin{bmatrix}0 & 0 \\ -2 & -2\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix}$$

$$\boxed{\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & -2\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix}}$$

$\longleftarrow$ Stabilized system has eigs $\lambda = -1, -1$.

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

Case 1: $\quad \underline{u}(t) = 0 \quad$ and $\quad \underline{x}(0) = \underline{x}_0$.

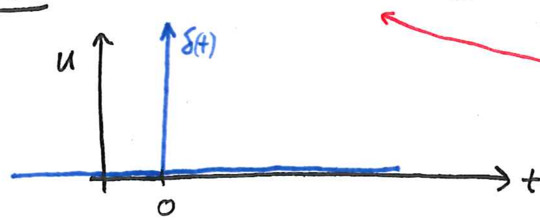
Same as $\quad \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}\,,\ \underline{x}(0) = \underline{x}_0 \quad$ (unforced system)

$$\boxed{\underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{x}_0} \longleftarrow$$

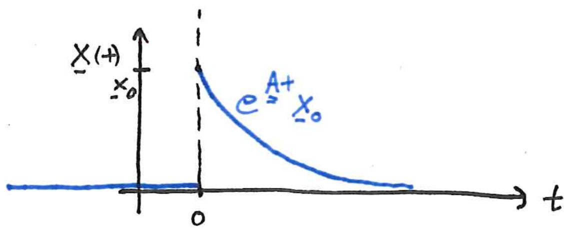
This is called an initial condition response

Case 2: $\quad u(t) = \delta(t)\,,\ \underline{\underline{B}} = \underline{x}_0\,,\ $ and $\ \underline{x}(0) = \underline{0}$

This is called an impulse response.



$$\underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{\underline{B}}$$

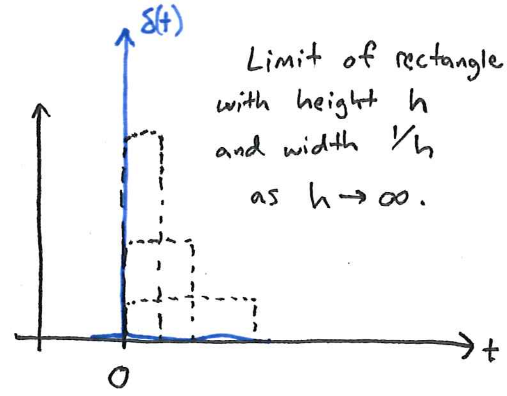


$$\Rightarrow \boxed{\underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{x}_0}$$

(See next page on "Impulse Response")

Impulse Response

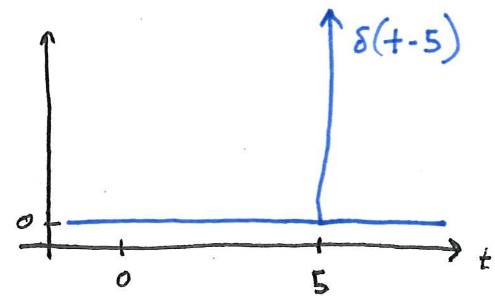
$\delta(t)$ - Dirac delta function



$$\int_{-\infty}^{\infty} \delta(t)\, dt = \underbrace{1}_{\text{unit area under } \delta(t)}$$

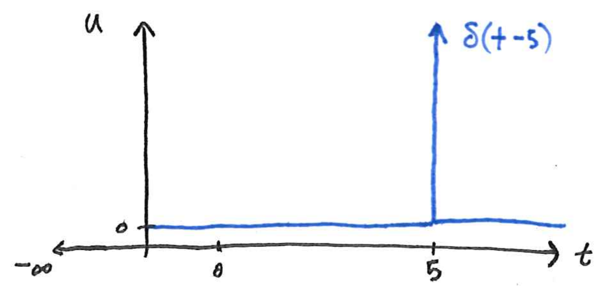
In fact, $\displaystyle\int_{-\varepsilon}^{+\varepsilon} \delta(t)\, dt = 1$ for all $\varepsilon > 0$.

Delta function delayed "τ" time in the future : $\delta(t-\tau)$.



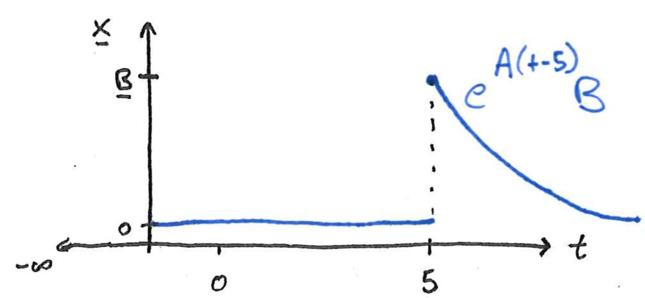
Example:

$$\begin{cases} \dot{\underline{x}} = \underline{\underline{A}}\, \underline{x} + \underline{\underline{B}}\, \underline{u} \\ \underline{x}(0) = 0 \\ u = \delta(t-5) \end{cases}$$



$$\underline{x}(t) = \underline{x}(0) + \int_0^t \left[\underline{\underline{A}}\, \underline{x}(\tau) + \underline{\underline{B}}\, \underline{u}(\tau)\right] d\tau$$

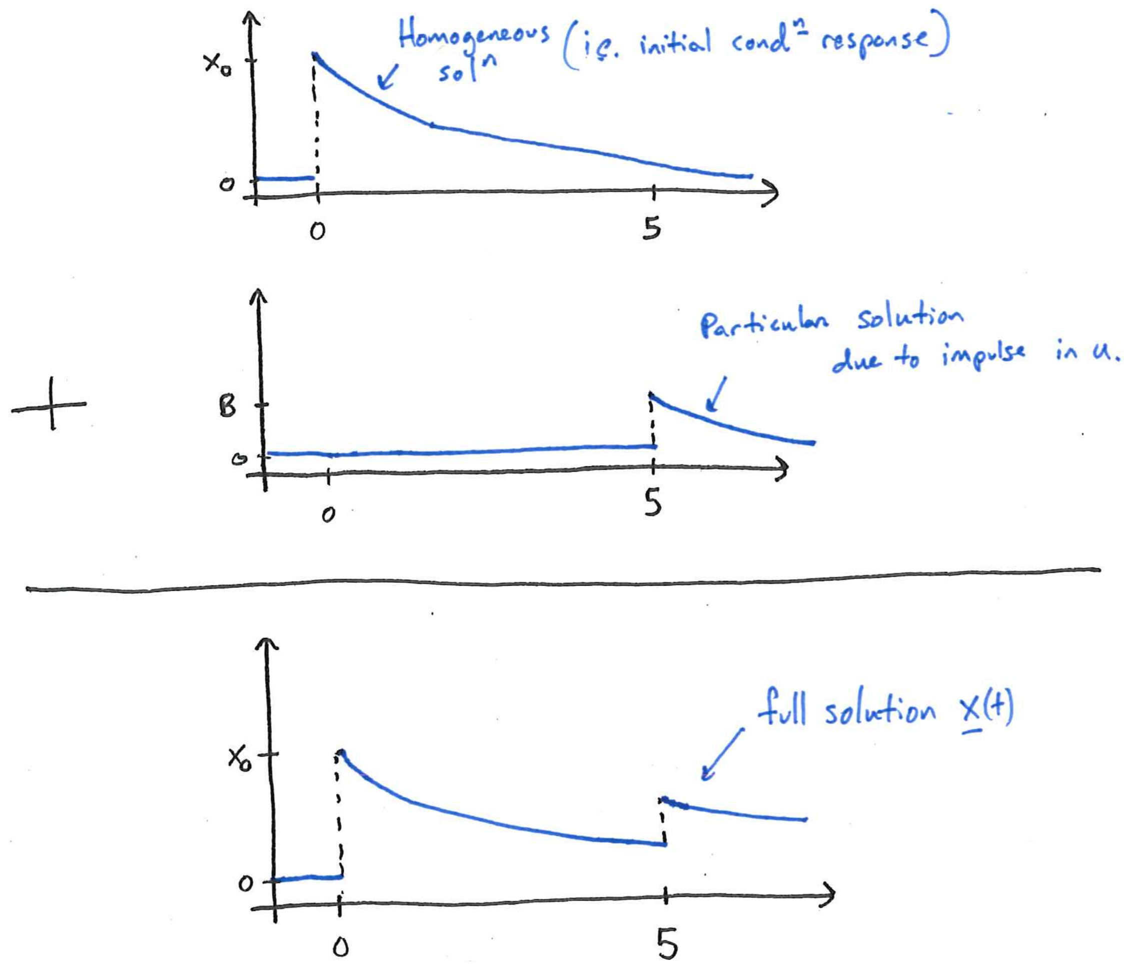
$$= \begin{cases} 0 & t < 5 \\ B & t = 5^+ \\ e^{A(t-5)} B & t > 5 \end{cases}$$



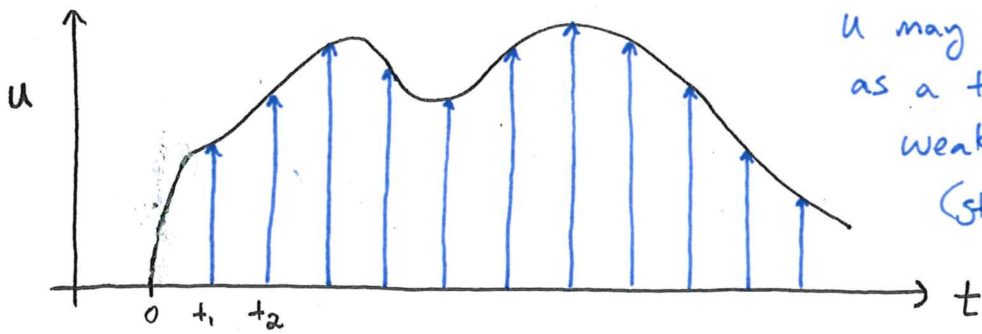
Case 3:

$u(t) = \delta(t-5)$, $\underline{\underline{B}} = B$, $\underline{x}(0) = \underline{x}_0$

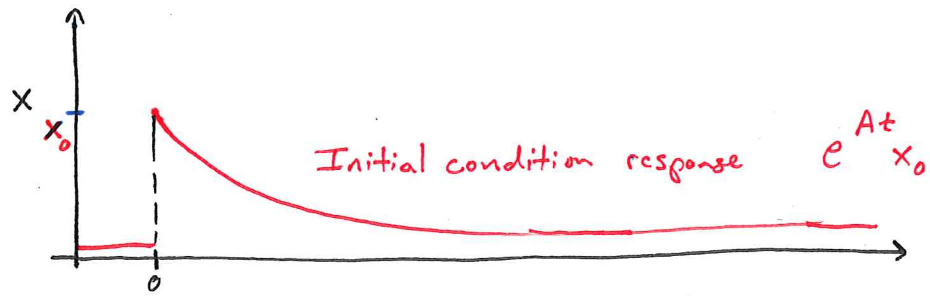
i.e. initial condition <u>and</u> impulsive input



Generic Input: Convolution!



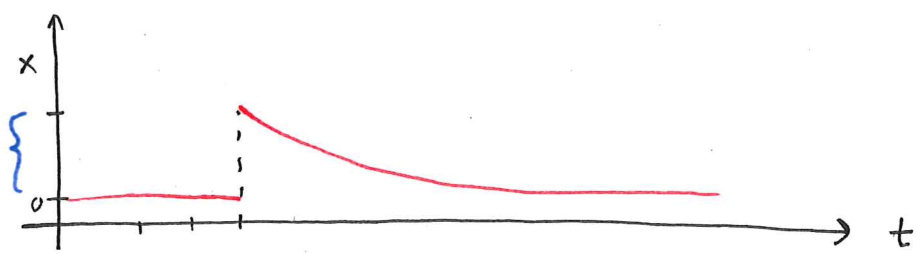
u may be thought of as a train of very weak impulses (strength $u(t)dt$)



$(t_1)dt$ strength ↑

+

$u(t_2)dt$ strength ↑



+

+ ... forever

$$\boxed{\underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{x}(0) + \underbrace{\int_0^t e^{\underline{\underline{A}}(t-\tau)}u(\tau)\,d\tau}_{\text{convolution integral}}}$$

Call $\underbrace{e^{At}}_{\text{impulse response}} = h(t) \Longrightarrow \int_0^t e^{A(t-\tau)}u(\tau)\,d\tau \triangleq \underline{\underline{h * u}}$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \implies \underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{x}(0) + \int_0^t e^{\underline{\underline{A}}(t-\tau)}\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau$$

In Matlab, we may simulate systems of the form:

$$\left.\begin{aligned} \dot{\underline{x}} &= \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \leftarrow \text{input} \\ \text{output} \rightarrow \underline{y} &= \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u} \\ &\quad\ \ \uparrow \text{internal state} \end{aligned}\right\} \text{called state-space form...}$$

For our purposes, we want to simulate & measure the whole state, so $\underline{y} = \underline{x}$: $\underline{\underline{C}} = \underline{\underline{I}}$, $\underline{\underline{D}} = \underline{\underline{0}}$

$\uparrow$ identity

$$\ddot{\theta} = -\sin(\theta) + \tau \xrightarrow{\text{at } \theta = 0} \frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tau$$

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \underline{\underline{C}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{\underline{D}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATLAB COMMANDS:

```
sys = ss(A,B,C,D)
impulse (sys)
step (sys)
lsim (sys)
bode (sys)
nyquist (sys)
:
```

```
clear all, close all, clc

%% simulate forced pendulum in down position
A = [0 1; -1 -.1];  % added small damping (-.1 omega)
B = [0; 1];
C = eye(2);
D = [0; 0];

sys = ss(A,B,C,D);

%% impulse response
impulse(sys,100)

%% linear response to arbitrary input
t = 0:.01:50;
u = 0*t;
u(1001:2000) = (1:1000)/10000;
u(2001:3000) = (1000-(1:1000))/10000;
plot(t,u)
lsim(sys,u,t)
```