L12: October 22, 2014

ME 564, Fall 2014

Overview of Topics:

① Linear ODEs w/ Forcing

$$\ddot{x} + 3\dot{x} + 2x = f(t)$$

- Method 1: undetermined coefficients
- Method 2: variation of parameters (very powerful)

② What does "Linear" mean

- Linear superposition
- Convolution.

Differential equations with forcing

(forcing may be disturbance or control)

$(*) \qquad \ddot{x} + 3\dot{x} + 2x = 0 \qquad$ 'homogeneous'

$(**) \qquad \ddot{x} + 3\dot{x} + 2x = \underbrace{f(t)}_{\text{forcing}} \qquad$ 'inhomogeneous'

Example: $f(t) = e^{-3t}$

Part 1: Solution to the $(*)$ is called the homogeneous sol^n

$$(*) \rightarrow \quad x(t) = k_1 e^{-t} + k_2 e^{-2t}$$

k_1, k_2 determined by initial conditions...

Part 2: Find particular sol^n to $(**)$ using method of undetermined coefficients:

Assume $x_p(t) = k e^{-3t} \Rightarrow \dot{x} = -3k e^{-3t}, \ \ddot{x} = 9k e^{-3t}$

$$\left[9k - 3^2 k + 2k\right] e^{-3t} = e^{-3t} \Longrightarrow \underline{\underline{k = 1/2}}$$

$$x_p = \frac{1}{2} e^{-3t}$$

Part 3: Add two sol^{ns}: $\quad x(t) = k_1 e^{-t} + k_2 e^{-2t} + \frac{1}{2} e^{-3t}$

[Only possible for LINEAR ODEs!]

Variation of Parameters (very general)

$$\ddot{x} + 3\dot{x} + 2x = 0$$

$$x(t) = k_1 e^{-t} + k_2 e^{-2t}$$

$$\Downarrow$$

$$(*) \quad \begin{cases} x(t) = u_1(t) e^{-t} + u_2(t) e^{-2t} \\ \dot{x} = -u_1 e^{-t} - 2u_2 e^{-2t} \boxed{\underbrace{+ \dot{u}_1 e^{-t} + \dot{u}_2 e^{-2t}}_{\text{Set } = 0}} \\ \ddot{x} = u_1 e^{-t} + 4u_2 e^{-2t} - \dot{u}_1 e^{-t} - 2\dot{u}_2 e^{-2t} = (u_1 - \dot{u}_1)e^{-t} + (4u_2 - 2\dot{u}_2)e^{-2t} \end{cases}$$

condition 1

(Big assumption?)

Now, to solve forced eqn $\ddot{x} + 3\dot{x} + 2x = e^{-3t}$

plug in $(*)$:

$$\ddot{x} + 3\dot{x} + 2x = \Big[\overbrace{u_1 - 3u_1 + 2u_1}^{=0}\Big]e^{-t} + \Big[\overbrace{4u_2 - 6u_2 + 2u_2}^{=0}\Big]e^{-2t}$$

$$\boxed{+ \left[-\dot{u}_1\right]e^{-t} + \left[-2\dot{u}_2\right]e^{-2t} = \underbrace{e^{-3t}}_{\text{forcing}}}$$

condition 2

$$\dot{u}_1 e^{-t} + \dot{u}_2 e^{-2t} = 0 \implies \dot{u}_2 = -\dot{u}_1 e^{t}$$

$$-\dot{u}_1 e^{-t} - 2\dot{u}_2 e^{-2t} = e^{-3t} \implies \dot{u}_1 \big[\underbrace{-e^{-t} + 2e^{-t}}_{e^{-t}}\big] = e^{-3t}$$

$$\begin{cases} \dot{u}_1 = e^{-2t} \implies u_1 = -\frac{1}{2}e^{-2t} + k_1 \\ \dot{u}_2 = -\dot{u}_1 e^{t} \\ \quad = -e^{-t} \implies u_2 = e^{-t} + k_2 \end{cases}$$

Finally!

$$x(t) = \left[-\tfrac{1}{2}e^{-2t} + k_1\right]e^{-t} + \left[e^{-t} + k_2\right]e^{-2t}$$

$$\boxed{x(t) = \frac{1}{2}e^{-3t} + k_1 e^{-t} + k_2 e^{-2t}}$$

Linear ODE: $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$

In the absence of initial conditions, there may be many "solutions" (in fact, one for each $\lambda_i = \text{eig}(A)$).

Just as in $\ddot{x} + 3\dot{x} + 2x = 0$, both $x_1(t) = e^{-t}$ and $x_2(t) = e^{-2t}$ are solutions.

For a linear system, if $\underline{x}_1$ and $\underline{x}_2$ are both solutions, then $k_1 x_1 + k_2 x_2$ is a solution for any real k_1 or k_2 (or complex k_1, k_2!):

First, $\frac{d}{dt}\left(k_1 \underline{x}_1 + k_2 \underline{x}_2\right) = k_1 \dot{\underline{x}}_1 + k_2 \dot{\underline{x}}_2$

Next, $\underline{\underline{A}}\left(k_1 \underline{x}_1 + k_2 \underline{x}_2\right) = k_1 A \underline{x}_1 + k_2 \underline{\underline{A}}\,\underline{x}_2$

So $\underline{x} = k_1 \underline{x}_1 + k_2 \underline{x}_2$ is a solution!

This is called the superposition principle.