# L11: Oct. 20, 2014 	ME564, Fall 2014

Overview of Topics:

① Nearly degenerate solutions to $\dot{\underline{x}} = A\underline{x}$

i.e. nearly parallel eigenvectors $\rightarrow$ $x(t) = te^{\lambda t}$

② Difference between $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

and $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

③ Examples: Shear flow

and bypass transition to turbulence ...

(Nearly) degenerate solutions to $\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$

i.e. nearly parallel eigenvectors of $\underline{\underline{A}}$...

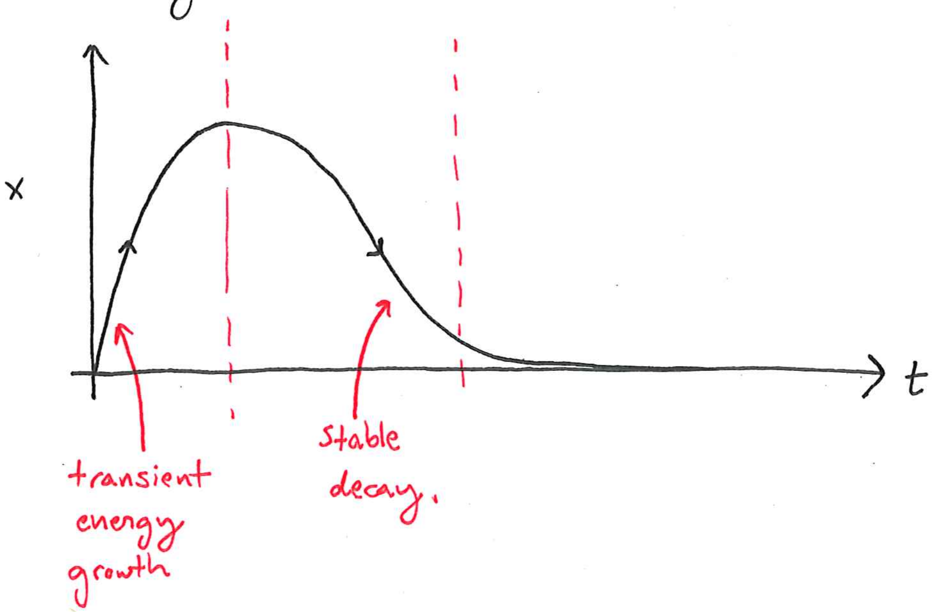
Stable system!!

$$A = \begin{bmatrix} -0.009 & 1 \\ 0 & -0.01 \end{bmatrix} \qquad \boxed{\begin{aligned} \lambda_1 &= -0.01 \\ \lambda_2 &= -0.009 \end{aligned}} \qquad \left(\text{about } 10\% \text{ different}\right)$$

$$\underline{\underline{\xi_1}} : \big[A - \lambda_1 I\big]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .001 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \xi_1 = \begin{bmatrix} 1 \\ -.001 \end{bmatrix}$$

$$\underline{\underline{\xi_2}} : \big[A - \lambda_2 I\big]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -.001 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \xi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Plot solution in MATLAB (PPLANE)
- Simulate using ode45



```
clear all, close all, clc

t = 0:.1:1000;
A = [-.009 1; 0 -.01];
y0 = [0; 1];
[t,y] = ode45(@(t,y)A*y, t, y0);
plot(t,y)
legend('x','v')
ylabel('x, v')
xlabel('Time')

%%
t = 0:.01:20;
A = [-1 1; 0 -1];
[t,y] = ode45(@(t,y)A*y, t, y0);
plot(t,y)
xlabel('Time')
ylabel('x,v')
legend('x','v')
```

Final case: $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ where $\underline{\underline{A}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

First, a proposition: If $ST = TS$ (not true in general)

then $e^{S+T} = e^S e^T$.

To prove, use binomial theorem: $(S+T)^n = n! \sum_{j+k=n} \frac{S^j T^k}{j!\,k!}$ ......

Check: $A = \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{S} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{T}$ $\qquad ST = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} = TS$ ✓

$$\boxed{e^{At} = e^{St} e^{Tt} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}}$$

new amazing, term!

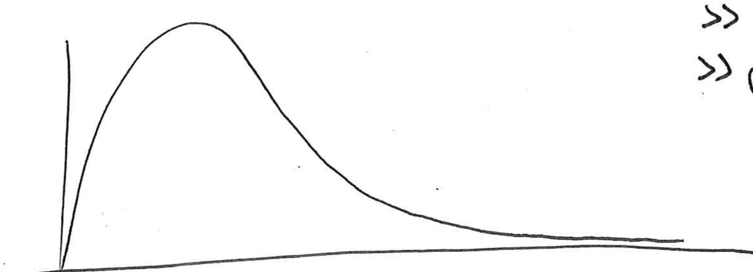
$$e^{St} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}$$

$$e^{Tt} = I + Tt + \underbrace{\frac{T^2 t^2}{2!} + \cdots}_{\text{All} = 0!}$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$te^{\lambda t}$ is called a secular term.

```
>> t=0:.01:20;
>> plot(t,t.*exp(-t))
```



If three repeated eigenvalues,

possible to get $t^2 e^{\lambda t}$ terms...

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \longrightarrow e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

(unverified...)

A tale of two '$\underline{\underline{A}}$' matrices ...

Case 1

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \left.\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \end{matrix}\right\} \text{repeated eigenvalue!}$$

Eigenvectors are $\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$[A - \lambda I]\underline{\xi} = \underline{0}$ ... note that $A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

has zero rank, so $\underline{\underline{A}}\underline{x} = \underline{0}$ for all $\underline{x}$.

Case 2

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \left.\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \end{matrix}\right\} \text{repeated eigs!}$$

$[A - \lambda I] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 1 ... so only 1 eigenvector...

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\underline{\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}}$$

To find second '<u>generalized eigenvector</u>'

solve $\boxed{[\underline{\underline{A}} - \lambda \underline{\underline{I}}]^2 \underline{\xi}_2 = \underline{0}}$

The difference is the rank of $[\underline{\underline{A}} - \lambda \underline{\underline{I}}]$!

$$\exp\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} t\right) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \qquad \exp\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t\right) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

extra terms!

Diagonalization of $\underline{\underline{A}}$

If eigenvalues λ of A are <u>real</u> and <u>distinct</u>

then eigenvectors T span $\mathbb{R}^n$ (n-dimensional real vector space)

and
$$T^{-1}AT = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

For a real-valued $\underline{\underline{A}}$,

any complex eigs must come in pairs

$$\lambda \pm i\omega \qquad T^{-1}AT = \underbrace{\begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix}}_{J} \rightarrow e^{Jt} = \begin{bmatrix} \cos & \sin \\ -\sin & \cos \end{bmatrix}$$

For repeated real eigenvalues of $\underline{\underline{A}}$

<u>Case 1</u>

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1, 1$

$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

<u>Case 2</u>

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1, 1$

$[A - \lambda I] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[A - \lambda I]^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (chosen perpendicular to ξ_1)

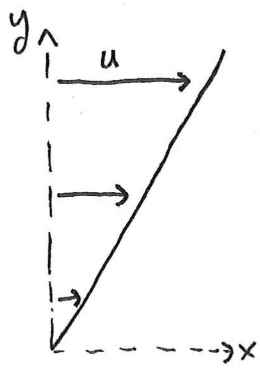
In general:
$$A = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

rank$(A - \lambda I) = 2$ rank$(A - \lambda I) = 1$ rank$(A - \lambda I) = 0$

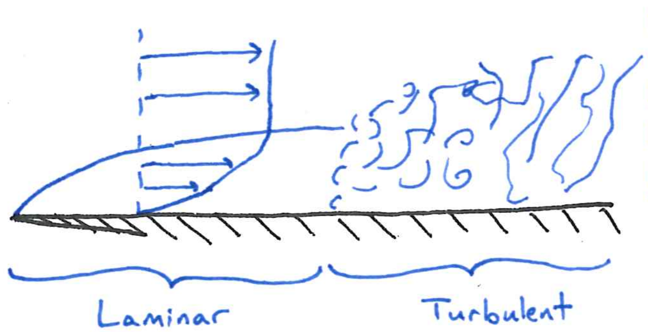
Examples in Fluid Flow:

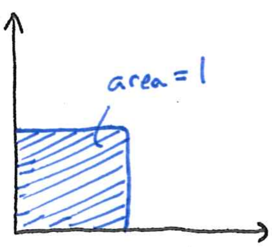
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is 'non-normal' : i.e. $A^T A \neq A A^T$ (check this!)

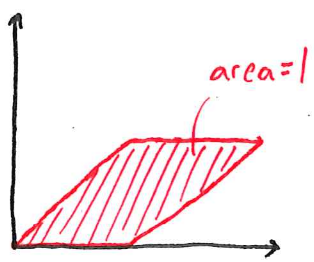
This is especially common in shear dominated flows:



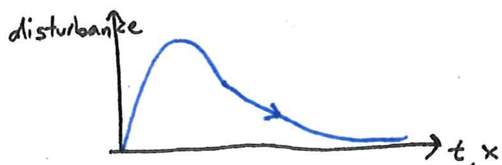
$$\frac{\partial u}{\partial y} = k.$$







In "Laminar" regime, equations are linearly stable



However if transient growth is large enough, it may trip system to turbulent by exciting nonlinearities.