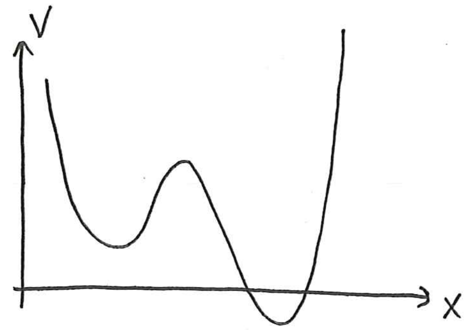
Particle in a potential well and linearizing nonlinear ODEs

Consider a potential $V(x)$ and imagine dropping a particle "bead" x that can roll down this surface.



The force is $F = -\frac{\partial V}{\partial x}$

so Newtons 2^{nd}: $\boxed{\ddot{x} = -\frac{\partial V}{\partial x}}$

Alternatively, we may use the Lagrangian:

kinetic energy: $T(x,\dot{x}) = \frac{1}{2}\dot{x}^2$

potential energy: $V(x)$

Lagrangian: $L(x,\dot{x}) = T(\dot{x}) - V(x)$

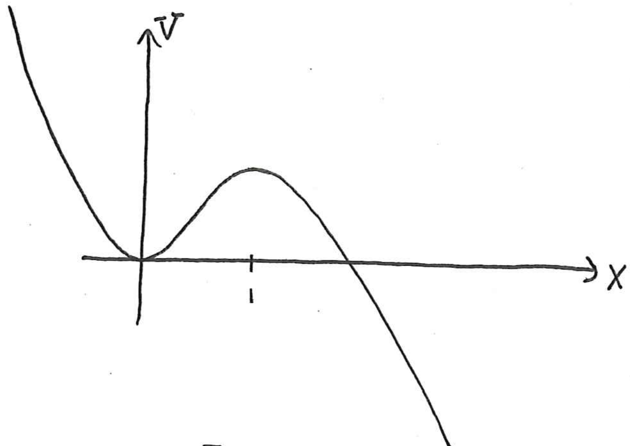
$= \frac{1}{2}\dot{x}^2 - V(x)$

Euler-Lagrange Equations: $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0}$

Super useful!! Among the most general expressions in all of physics! Encompasses classical and quantum mechanics special and general relativity!!

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{d}{dt}\dot{x} = \ddot{x} \\ \frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x} \end{aligned}\right\} \Longrightarrow \boxed{\ddot{x} = -\frac{\partial V}{\partial x}} \checkmark$$

Ex: $\ddot{x} = -x + x^2$ $\left(\text{i.e. } V(x) = \frac{x^2}{2} - \frac{x^3}{3}\right)$



$\dot{x} = v$

$\dot{v} = -x + x^2$

Fixed points for $v = 0$

and $x = 0$ or $x = 1$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underline{f}\left(\begin{bmatrix} x \\ v \end{bmatrix}\right) = \begin{bmatrix} f_1(x,v) \\ f_2(x,v) \end{bmatrix} \Longrightarrow \frac{D\underline{f}}{D\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1+2x & 0 \end{bmatrix}$$

First fixed point:

$$\frac{Df}{D\underline{x}}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

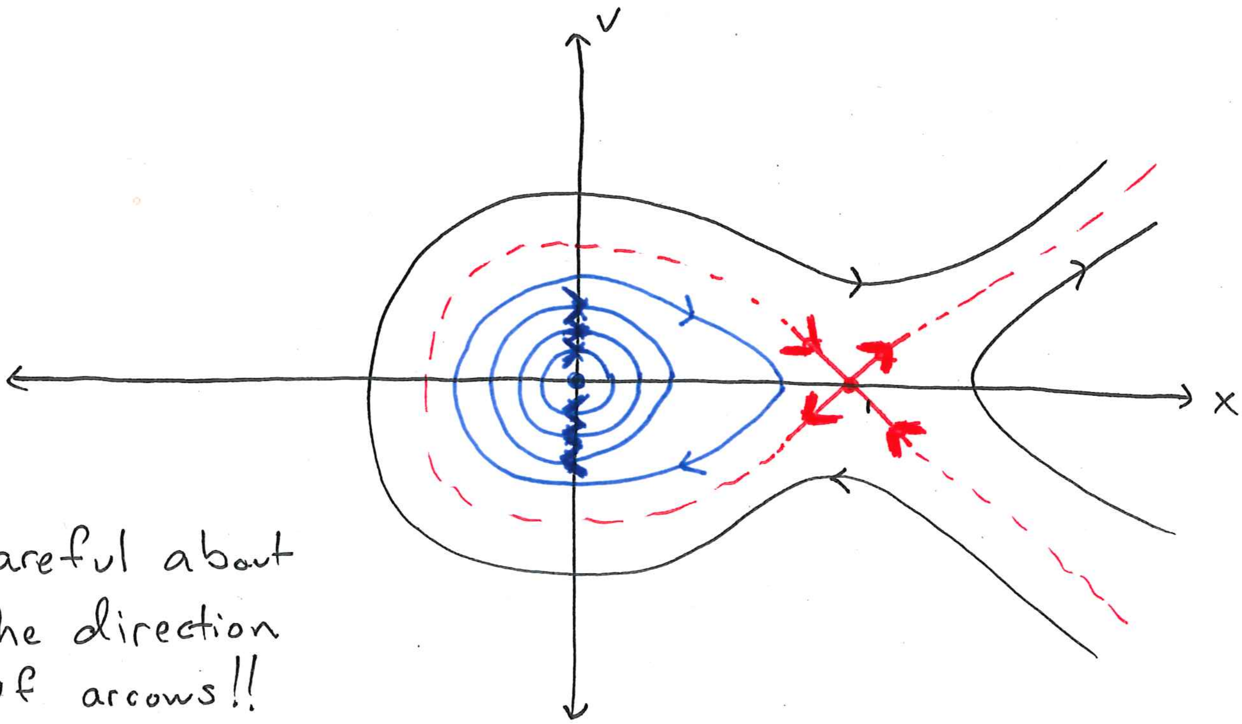
$\lambda = \pm i$ (center)

Second fixed point:

$$\frac{Df}{Dx}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda = \pm 1$ (saddle)

$$\underbrace{\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{Verify}}$$



careful about the direction of arrows!!

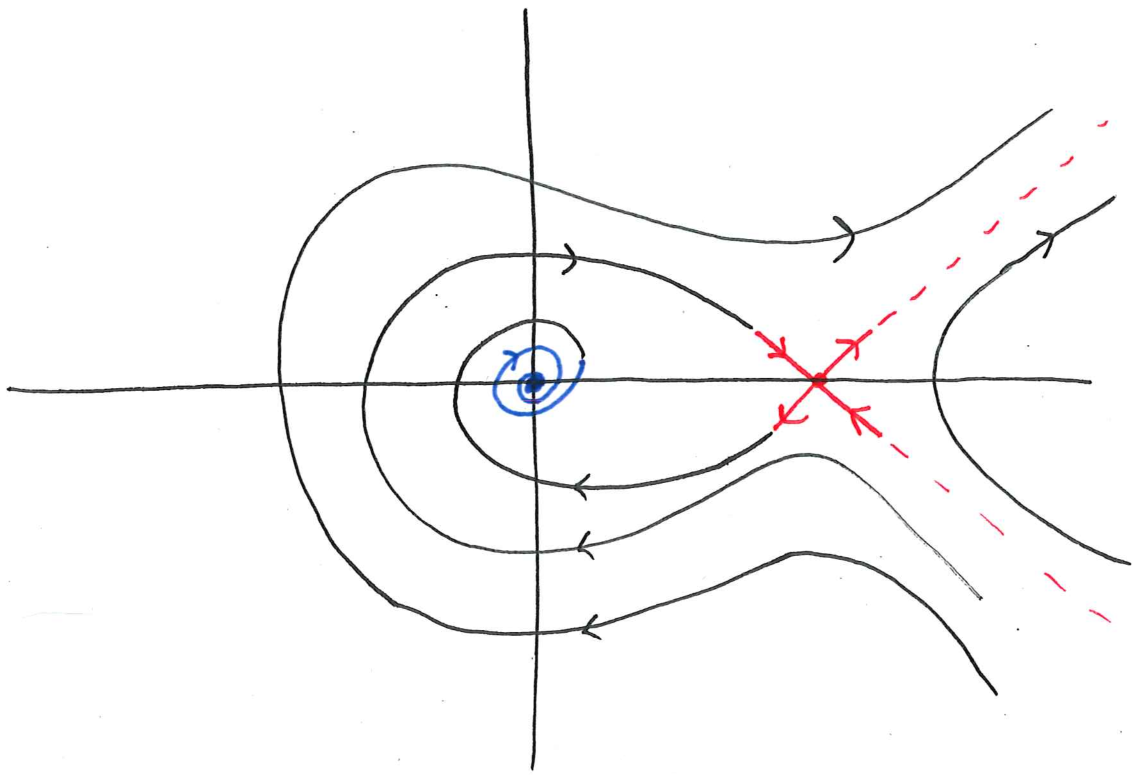
try this in "pplane"

Ex cont^{d}: If we add damping ...

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -x + x^2 - v\end{aligned} \implies \frac{Df}{Dx} = \begin{bmatrix} 0 & 1 \\ -1+2x & -1 \end{bmatrix}$$

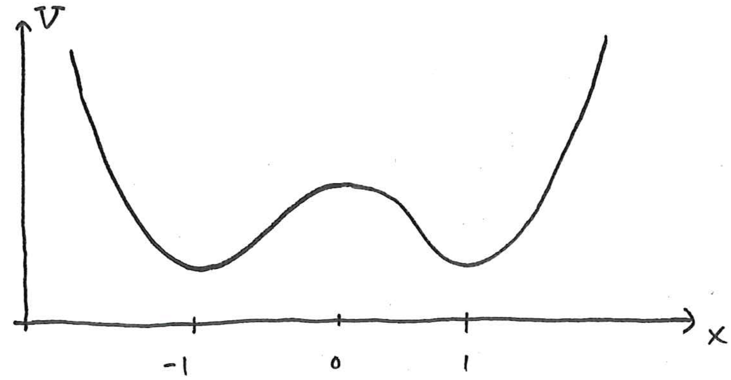
$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ becomes a spiral sink

$\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ still a saddle



$\underline{\text{Ex}}$ $\quad \ddot{x} = -x^3 + x \underbrace{- \dot{x}}_{\text{optional damping}}$ $\quad \left(\text{i.e. } V(x) = \frac{x^4}{4} - \frac{x^2}{2} \right)$

$$\dot{x} = y$$
$$\dot{y} = +x - x^3 \underbrace{- y}_{\text{opt. damping}}$$



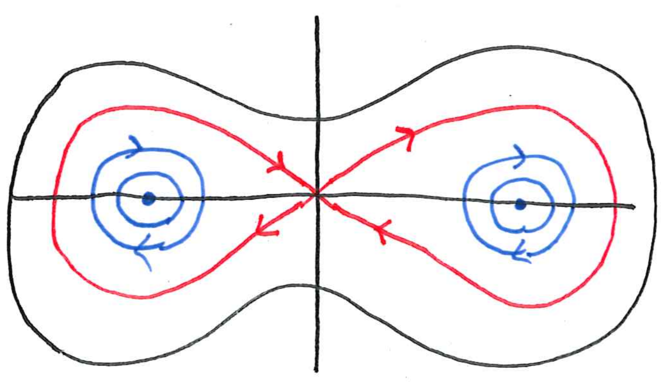
Fixed Points : $\quad v = 0$ and $x = 0$ or ± 1 $\Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\frac{Df}{D\underline{x}} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -1 \end{bmatrix} \Longrightarrow \frac{Df}{Dx}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \frac{Df}{Dx}\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

saddle

center
(or spiral sink w/ damping)

Without Damping



With Damping