L09: Oct. 15, 2014 ME 564, Fall 2014

Overview of Topics:

1. Examples of $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$
   - center: $\lambda_{\pm} = \pm i$
   - Stable spiral
2. Linearization
   - 1D example: Logistic equation.
   - 2D examples: $\ddot{x} = -\frac{\partial V}{\partial x}$ where $V(x)$ is potential.

Linearizing a Nonlinear System at a Fixed Point

2D Example

$\dot{x} = f(x)$ $\qquad$ $\bar{x}$ is a fixed point if $f(\bar{x}) = 0$!

(i.e. $\dot{\bar{x}} = 0$)

For x near $\bar{x}$, so $\Delta x = x - \bar{x}$ is small,

$$\dot{x} = f(x) = \underbrace{f(\bar{x})}_{\substack{=0 \\ \text{for f.p.}}} + \left.\frac{Df}{Dx}\right|_{x=\bar{x}} \cdot (x-\bar{x}) + \underbrace{\left.\frac{D^2f}{Dx^2}\right|_{x=\bar{x}} \cdot (x-\bar{x})^2 + \left.\frac{D^3f}{Dx^3}\right|_{x=\bar{x}} \cdot (x-\bar{x})^3 + \cdots}_{\substack{\text{small for} \\ \Delta x \text{ small!}}}$$

$$\dot{x} \approx \left.\frac{Df}{Dx}\right|_{x=\bar{x}} \cdot (x-\bar{x}) \implies \dot{x} - \dot{\bar{x}} = \boxed{\frac{d}{dt}\Delta x = \left.\frac{Df}{Dx}\right|_{x=\bar{x}} \cdot \Delta x}$$ Linear!!

For a vector valued $\underline{f}(\underline{x})$, $\frac{Df}{Dx}$ is a matrix!

Example:
$$\underline{f}(\underline{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1 - x_1^2 \\ x_1 + x_2 \end{bmatrix}$$

(Fixed Pts: $x_1 = 0 \quad x_2 = 0$
$x_1 = 1 \quad x_2 = -1$)

$$\frac{D\underline{f}}{D\underline{x}} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1-2x_1 & 0 \\ 1 & 1 \end{bmatrix}$$

FP1

$$\bar{\underline{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{Df}{Dx}(\bar{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\lambda = 1, 1$ Unstable Source!

FP2

$$\bar{\underline{x}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \frac{Df}{Dx}(\bar{x}) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\lambda = \pm 1$ SADDLE!

Fixed Points and Linearization of Nonlinear Systems

1D Example

$$(*) \quad \dot{x} = f(x) = \underset{\text{population}}{x}\,\underbrace{(\overset{\text{max population}}{P_{max}} - x)}_{\text{resources}}$$

Logistic equation for population growth

First, find fixed points where

$$\dot{x} = f(\bar{x}) = 0 \implies \bar{x} = 0 \text{ and } \bar{x} = P_{max}$$

Near $\bar{x}$, we may linearize the dynamics:

$$\frac{d}{dt}(x - \bar{x}) = \dot{x} = f(x) = \underbrace{\cancelto{0}{f(\bar{x})}^{\text{fixed pt.}} + \frac{Df}{Dx}(\bar{x})\cdot(x-\bar{x}) + \frac{D^2 f}{Dx^2}(\bar{x})\cdot(x-\bar{x})^2 + \ldots}_{\text{Taylor expand about fixed point...}}$$

For small $\Delta x = x - \bar{x} \approx 0$,

Dynamics are approximately linear!

$$\boxed{\frac{d}{dt}\Delta x = \frac{Df}{Dx}(\bar{x})\cdot(x - \bar{x})}$$

For $(*)$: $\frac{Df}{Dx} = P_{max} - 2x$

Case 1: $\bar{x} = 0$

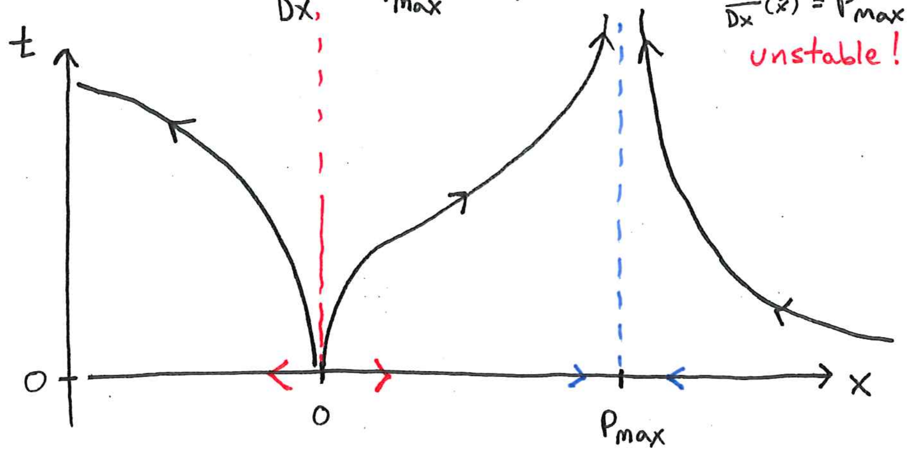
$\frac{Df}{Dx}(\bar{x}) = P_{max}$

unstable!

Case 2: $\bar{x} = P_{max}$

$\frac{Df}{Dx}(\bar{x}) = -P_{max}$

stable!



Example $\qquad \dfrac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$

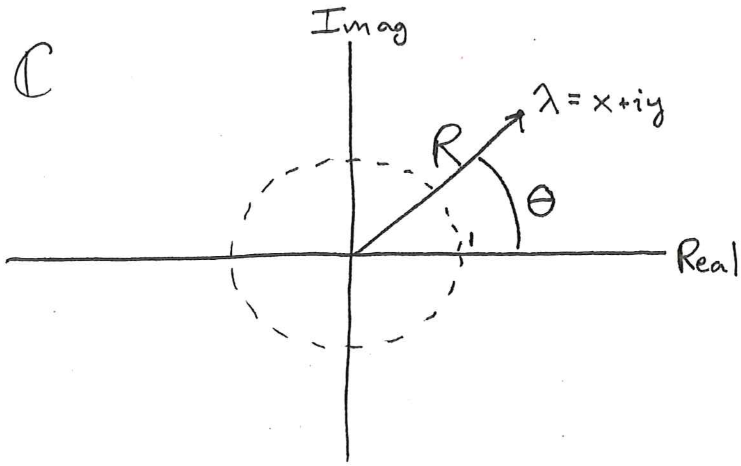
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 4 = 0 \implies \boxed{\lambda = \pm 2i}$$

imaginary!

More soon...

$\mathbb{C}$



$x = R\cos(\theta)$

$y = R\sin(\theta)$

$$\lambda = (R, \theta) \qquad = Re^{i\theta}$$

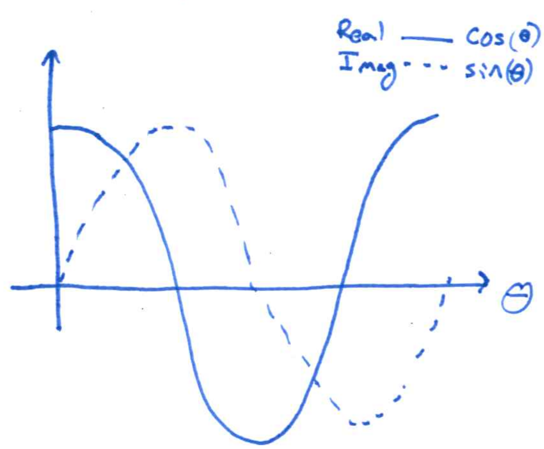
$$\lambda^2 = (R^2, 2\theta) \qquad = R^2 e^{2i\theta}$$

$$\vdots$$

$$\lambda^N = (R^N, N\theta) \qquad = R^N e^{iN\theta}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$



Example $\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x}$ $\qquad \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$

$$\det(A - \lambda I) = \lambda^2 + 4 = 0 \Rightarrow \boxed{\lambda = \pm 2i} \quad \text{imaginary!}$$

ξ_1 for $\lambda_1 = 2i$: $\quad A - 2iI = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2ix = 2v \\ 2x = -2iv \end{matrix} \Rightarrow \begin{matrix} x = 1 \\ v = i \end{matrix} \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

ξ_2 for $\lambda_2 = -2i$: $\quad A + 2iI = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}$

$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2ix = -2v \\ 2x = 2iv \end{matrix} \Rightarrow \begin{matrix} x = 1 \\ v = -i \end{matrix} \Rightarrow \xi_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \Rightarrow T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

← verify this by hand!! (also check $TDT^{-1} = A$)

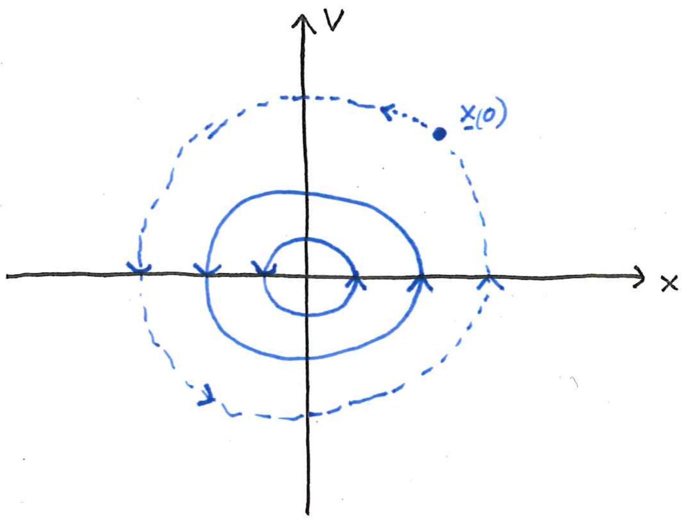
$$\underline{x}(t) = T e^{\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} t} T^{-1} \underline{x}(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \underline{x}(0)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{2it} & -ie^{2it} \\ e^{-2it} & ie^{-2it} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2it} + e^{-2it} & i(-e^{2it} + e^{-2it}) \\ i(e^{2it} - e^{-2it}) & e^{2it} + e^{2it} \end{bmatrix} \underline{x}(0)$$

$$= \underbrace{\frac{1}{2} \begin{bmatrix} 2\cos(2t) & 2\sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}}_{\text{Real Valued!!}} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

Ex. Con^{td}:
$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}}_{\text{Pure Rotation Matrix...}}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}} \leftarrow \text{Solution!}$$



$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \implies \ddot{x} + 4x = 0$

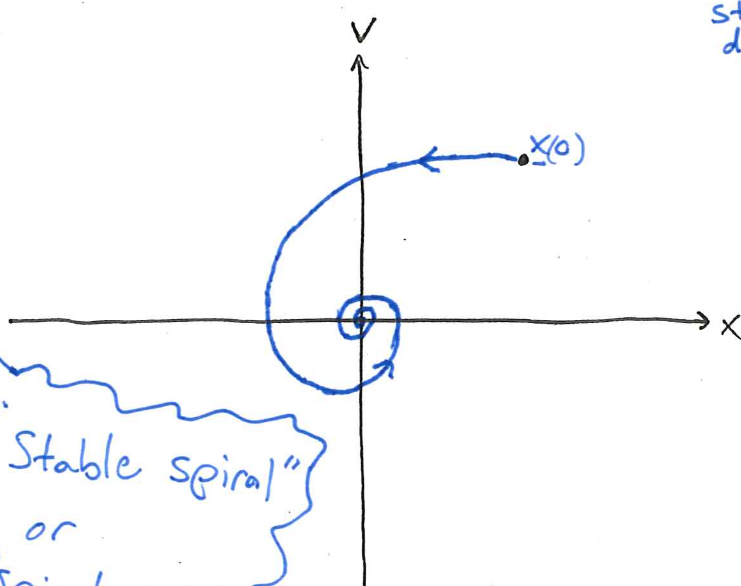
Undamped spring-mass...

"center" fixed point

– marginally stable...

Ex. $\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \implies \lambda = -1 \pm 2i$

Solutions look like $e^{\lambda t} = \underbrace{e^{-t}}_{\text{stable decay}}\big[\underbrace{\cos(2t) + i\sin(2t)}_{\text{oscillation}}\big]$



"Stable spiral" or "Spiral sink"