Overview of Topics :

1. Eigenvalues & eigenvectors to diagonalize $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$
2. Geometry of e-vals, e-vecs
3. Evals & Evecs in general
4. Examples
5. Solution to $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$

Recall: To "diagonalize" $\dot{x} = Ax$

i.e. to change coordinates $x = Tz$

so that dynamics are diagonal: $\dot{z} = Dz$

We need to solve the

eigen value equation:

$$\boxed{AT = TD}$$

columns of T are eigenvectors

diagonal entries of D are eigenvalues.

Eigenvalues & Eigenvectors

'Eigen' = latent or characteristic

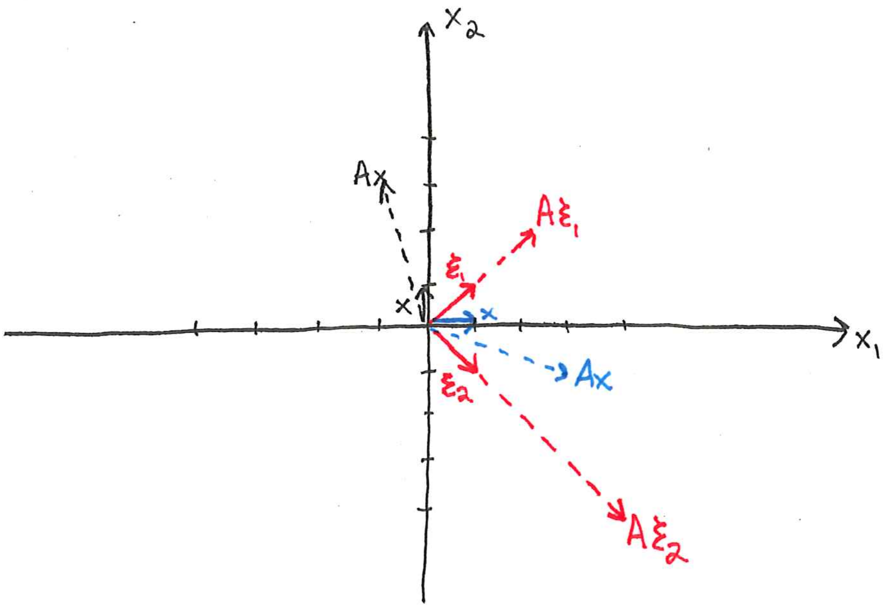
$$\boxed{Ax = \lambda x \quad \text{for special vectors } x \text{ and special values } \lambda.}$$

Eigenvalue eq$^{\underline{n}}$ for single eigen pair (x, λ).

Example : $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

try $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow A\underline{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

try $\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow A\underline{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$



Special $\underline{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow A\underline{\xi}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\lambda_1 = 2$

Special $\underline{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow A\xi_2 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

$\lambda_2 = 4$

$$\left[\begin{array}{l} \text{Exactly 2 eigenvalues } \lambda_1, \lambda_2 \\ \quad \text{and 2 eigenvectors } \xi_1, \xi_2 . \end{array} \right]$$

Eigenvalues & Eigenvectors in general:

$$Ax = \lambda x = \lambda I x$$

identity matrix

$$(A - \lambda I)\underline{x} = \underline{0}$$

<u>Case 1</u>: $\underline{x} = \underline{0}$ (not interesting)

<u>Case 2</u>: $\underline{x} \neq \underline{0}$ and $\det(A - \lambda I) = 0$

"$A - \lambda I$" is <u>singular</u>

meaning that it maps some vectors to $\underline{0}$.

$$\boxed{\det(A - \lambda I) = 0}$$

polynomial equation whose roots are eigenvalues!

Characteristic Equation

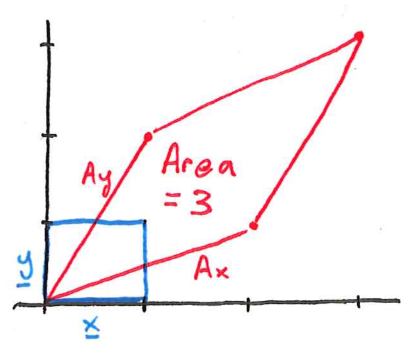
Remember 3×3 determinant...

$$\det\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = b_{11}\cdot\det\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} - b_{12}\cdot\det\begin{pmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{pmatrix} + b_{13}\cdot\det\begin{pmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

Determinant measures the volume of a unit cube after mapping through $\underline{A}$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det(A) = 4 - 1 = \underline{\underline{3}}$$



Example: $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \Longrightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}$

Step 1
Compute λ

$$\det(A - \lambda I) = (3-\lambda)^2 - 1$$
$$= \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2) = 0$$

Recall

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$\det(B) = b_{11} b_{22} - b_{12} b_{21}$

$\Longrightarrow$ $\boxed{\text{eigenvalues are } \lambda_1 = 2, \ \lambda_2 = 4.}$

Step 2
compute $\underline{x}$ given λ.

$\lambda_1 = 2$: $A - 2I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow x_1 = x_2$$

$$\boxed{\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \lambda_1 = 2}$$

Note $\xi = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \ldots$ also work...

$\lambda_2 = 4$: $A - 4I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow x_1 = -x_2$$

$$\boxed{\xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \lambda_2 = 4}$$

`>> [T, D] = eig(A)`

$$D = T^{-1}AT \implies A = TDT^{-1}$$

$$A^2 = (TD\underbrace{T^{-1})(T}_{T^{-1}T = I}DT^{-1})$$

$$= TD^2T^{-1}$$

$$A^3 = (TD\underbrace{T^{-1})(T}_{= I}D\underbrace{T^{-1})(T}_{= I}DT^{-1})$$

$$= TD^3T^{-1}$$

$$\vdots$$

$$A^N = TD^NT^{-1}$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots$$

$$= T\left[I + Dt + \frac{1}{2!}D^2t^2 + \frac{1}{3!}D^3t^3 + \ldots\right]T^{-1}$$

$$= Te^{Dt}T^{-1}$$

$$\implies \boxed{\begin{array}{c}\underline{x}(t) = Te^{Dt}T^{-1}\underline{x}(0) \\ \text{is solution to } \quad \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}.\end{array}}$$

>> [T, D] = eig(A)

$$T = \begin{bmatrix} | & | & & | \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ | & | & & | \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$D^N = \begin{bmatrix} \lambda_1^N & & & 0 \\ & \lambda_2^N & & \\ & & \ddots & \\ 0 & & & \end{bmatrix}$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}, \quad \underline{x}(0)$$

$$\Longrightarrow \quad \underline{x}(t) = \underbrace{T \underbrace{e^{Dt} \underbrace{T^{-1} \underline{x}(0)}_{z(0) \; ①}}_{z(t) \; ②}}_{x(t) \; ③}$$

① $x = Tz \Rightarrow z(0) = T^{-1}x(0)$ write IC in z-coords

② $z(t) = e^{Dt} z(0)$ solve $\dot{z} = Dz$

③ $x(t) = Tz(t)$ transform solution $z(t)$ back to $x(t)$...

Example $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

to compute T^{-1}: $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & .5 & .5 \\ 0 & -2 & -1 & 1 \end{array}\right]$

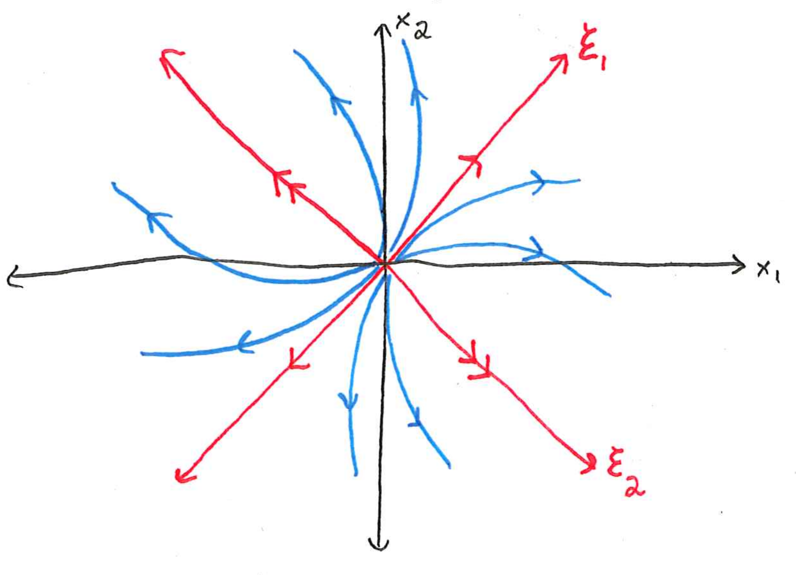
$$\left[\begin{array}{cc|cc} 1 & 0 & .5 & .5 \\ 0 & 1 & .5 & -.5 \end{array}\right] \Longrightarrow T^{-1} = \begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix}$$

In Matlab, check $T^{-1}AT = D$

$$\underline{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix} \underline{X}(0)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} .5e^{2t} & .5e^{2t} \\ .5e^{4t} & -.5e^{4t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

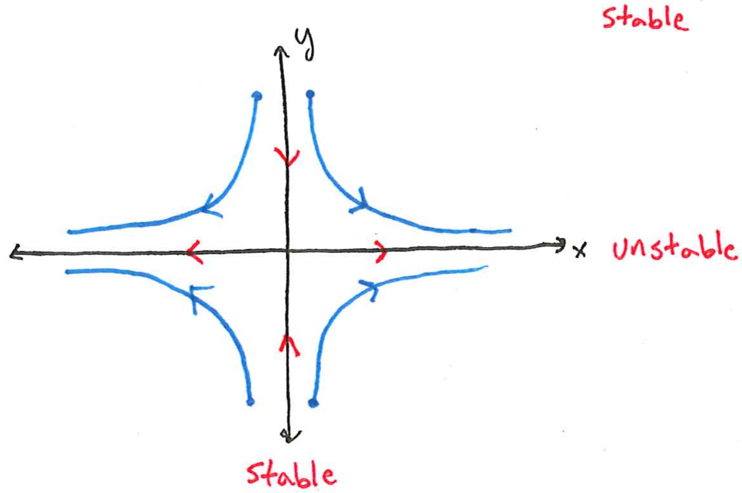
$$= \frac{1}{2} \begin{bmatrix} e^{2t}+e^{4t} & e^{2t}-e^{4t} \\ e^{2t}-e^{4t} & e^{2t}+e^{4t} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}$$



Try "pplane" in Matlab

Example:
$$\frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \Longrightarrow \begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}\overbrace{e^{t}}^{\text{unstable}} & 0\\ 0 & \underbrace{e^{-t}}_{\text{Stable}}\end{bmatrix}\begin{bmatrix}x(0)\\ y(0)\end{bmatrix}$$

Already diagonal!



Saddle point

Example

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 4 = 0 \implies \boxed{\lambda = \pm 2i}$$

imaginary!

More soon...

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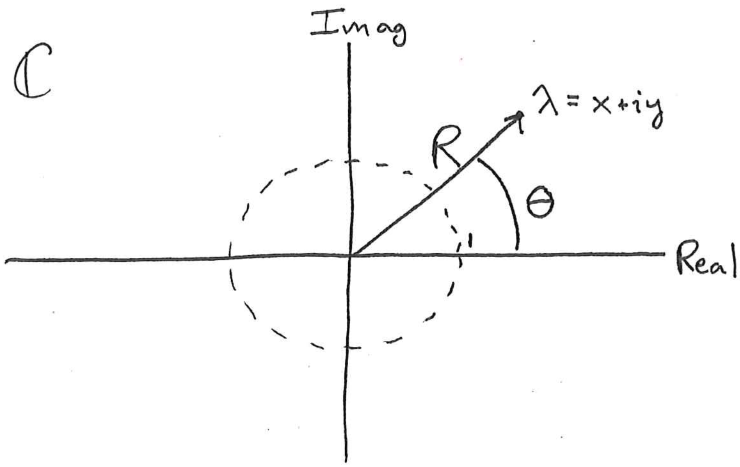

$$x = R\cos(\theta)$$
$$y = R\sin(\theta)$$

$$\lambda = (R, \theta) \qquad = Re^{i\theta}$$
$$\lambda^2 = (R^2, 2\theta) \qquad = R^2 e^{2i\theta}$$
$$\vdots$$
$$\lambda^N = (R^N, N\theta) \qquad = R^N e^{iN\theta}$$