<u>L06</u> : Oct. 8, 2014 ME 564, Fall 2014

Overview of Topics :

① Review : - Higher order systems
- characteristic equation
- matrix system of ODEs $\dot{y} = \underline{\underline{A}}\, y$

② Example : $\dot{y} = \underline{\underline{A}}\, y$
- eigenvalues of $\underline{\underline{A}}$ are roots of characteristic polynomial!!

③ Special matrix system of ODEs
$\dot{y} = \underline{\underline{D}}\, y$ where $\underline{\underline{D}}$ is Diagonal!

④ Derive eigenvalve equation to turn <u>any</u> system $\dot{y} = \underline{\underline{A}}\, y$ into <u>diagonal</u> form!

Higher Order Systems:

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$$

- OR -

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

Try $x(t) = e^{\lambda t}$ (note $\frac{d^n x}{dt^n} = \lambda^n e^{\lambda t}$
$= \lambda^n x(t)$)

$$\left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda + a_0\right) \underbrace{x(t)}_{e^{\lambda t}} = 0$$

$$\Longrightarrow \boxed{a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0}$$

Characteristic Equation!

In general, n solutions: $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_n e^{\lambda_n t}}$$

Need n initial conditions to determine constants $\{c_k\}_{k=1}^{n}$

Usually, use $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix}$ to solve for $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$

Note: you can always divide everything by a_n in top equation...

High Order ODE $\Longrightarrow$ System of 1st Order ODEs

$$x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

Introduce new variables

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = \ddot{x}$$
$$x_4 = \dddot{x}$$
$$\vdots$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -\left[a_{n-1} x_n + a_{n-2} x_{n-1} + \dots + a_2 x_3 + a_1 x_2 + a_0 x_1\right]$$

negative sign!

As a matrix system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\Longrightarrow \boxed{\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}}$$

$$\ddot{x} + 3\dot{x} + 2x = 0 \quad ;$$

$$\Downarrow$$

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2x - 3v \end{aligned}\right\} \quad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$\dot{y} = \underline{\underline{A}}\, y$$

eigs of A are roots of characteristic polynomial!

$$\det(A - \lambda I) = 0$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\lambda I} = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \boxed{\lambda^2 + 3\lambda + 2 = 0}$$

characteristic polynomial!

Close connection between eigenvalues of $\underline{\underline{A}}$ and solutions to ODE $\dot{y} = \underline{\underline{A}}\, y$.

Claim: $\det(A - \lambda I) = 0$

is equal to characteristic equation.

Example: $$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -\lambda - a_2 \end{bmatrix}$$

$$= -\lambda\left[\lambda^2 + \lambda a_2 + a_1\right] - 1 \cdot \left[a_0\right] + 0$$

$$= \underbrace{-\lambda^3 - \lambda^2 a_2 - \lambda a_1 - a_0 = 0}$$

$$\Longrightarrow \boxed{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0}$$

Characteristic Equation

We want to solve general systems of equations:

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} .$$

Case 1: Uncoupled dynamics

Different animals in a zoo (separated populations)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{x}_1 = \lambda_1 x_1 \Rightarrow x_1 = e^{\lambda_1 t} x_1(0)$$
$$\dot{x}_2 = \lambda_2 x_2 \Rightarrow x_2 = e^{\lambda_2 t} x_2(0)$$
$$\vdots$$
$$\dot{x}_n = \lambda_n x_n \Rightarrow x_n = e^{\lambda_n t} x_n(0)$$

$$\Downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}(t) = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}}_{\underline{\underline{e^{At}}}}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}(t=0)$$

(matrix exponential easy for diagonal A...

For a generic system of first-order, linear ODEs:

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$$

We want a change of coordinates $\underline{x} = \underline{\underline{T}}\,\underline{z}$ that diagonalizes the ODE:

$$\underline{\dot{z}} = \underline{\underline{D}}\,\underline{z}$$, where $\underline{\underline{D}}$ is diagonal.

$$T\dot{z} = \dot{x} = Ax \implies T\dot{z} = ATz$$

$$\implies \dot{z} = \underbrace{T^{-1}AT}_{=D}\,z$$

So we want T so that

$$T^{-1}AT = D$$

$$\implies \boxed{AT = TD}$$ eigenvalue equation

eigenvector

eigenvalue

$$\begin{bmatrix} & & \\ & \underline{\underline{A}} & \\ & & \end{bmatrix} \begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & 0 \\ 0 & & \ddots \\ & & & d_n \end{bmatrix}$$

$$\boxed{\text{MATLAB: } \gg [T,D] = \mathrm{eig}(A);}$$

>> [T, D] = eig(A)

$$D = T^{-1}AT \implies A = TDT^{-1}$$

$$A^2 = (TD\underbrace{T^{-1})(T}_{T^{-1}T = I}DT^{-1})$$

$$= TD^2T^{-1}$$

$$A^3 = (TD\underbrace{T^{-1})(T}_{=I}D\underbrace{T^{-1})(T}_{=I}DT^{-1})$$

$$= TD^3T^{-1}$$

$$\vdots$$

$$A^N = TD^NT^{-1}$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

$$= T\left[I + Dt + \frac{1}{2!}D^2t^2 + \frac{1}{3!}D^3t^3 + \dots\right]T^{-1}$$

$$= Te^{Dt}T^{-1}$$

$$\implies \boxed{\begin{array}{c} \underline{x}(t) = Te^{Dt}T^{-1}\underline{x}(0) \\ \text{is solution to } \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}. \end{array}}$$