L05 : Oct. 3, 2014 ME564, Fall 2014

Overview of Topics:

0. More second order systems!
1. Higher order systems

   and the characteristic equation

   - no analytic closed-form solution for polynomials w/ order ≥ 5.

2. Matrix systems of first order ODEs

3. Special Case: uncoupled system.

$\Downarrow$

Next Week: Eigenvalues & Eigenvectors

to solve $\dot{\underline{x}} = \underline{\underline{A}}\, \underline{x}$.

Example:

$$\ddot{x} + 3\dot{x} + 2x = 0 \quad , \quad \left.\begin{aligned} x(0) &= 2 \\ \dot{x}(0) &= -3 \end{aligned}\right\} \text{ICs}$$

Try $x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$$\left[\lambda^2 + 3\lambda + 2\right] e^{\lambda t} = 0$$

Characteristic Equation (very important!)

$$\boxed{\lambda^2 + 3\lambda + 2 = 0} \Rightarrow (\lambda+2)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1$$
$$\lambda = -2.$$

$$x(t) = k_1 e^{-t} + k_2 e^{-2t} \Rightarrow \begin{cases} x(0) = k_1 + k_2 = 2 \\ \dot{x}(0) = -k_1 - 2k_2 = -3 \end{cases}$$

$$\Rightarrow \underline{\underline{k_1 = k_2 = 1}}.$$

$$\boxed{x(t) = e^{-t} + e^{-2t}}$$

Stable: solution $x \to 0$ as $t \to \infty$.

$$\ddot{x} + 3\dot{x} + 2x = 0$$

$$\Big\Downarrow \text{ suspend variables}$$

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2x - 3v \end{aligned}\right\} \quad \underbrace{\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}}_{\dot{y} = \underline{\underline{A}}\, y} \; ; \; \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

\>> ode45

\>> eig(A)

Ex 2: What about $\ddot{x} - 3\dot{x} + 2x = 0$? $\quad x(0) = 2, \; \dot{x}(0) = 3$

$$\lambda^2 - 3\lambda + 2 = 0 \Longrightarrow (\lambda - 2)(\lambda - 1) = 0$$

$$\Longrightarrow \lambda = 1, 2$$

$$\boxed{x(t) = e^{t} + e^{2t}}$$ Unstable! $x \to \infty$ as $t \to \infty$.

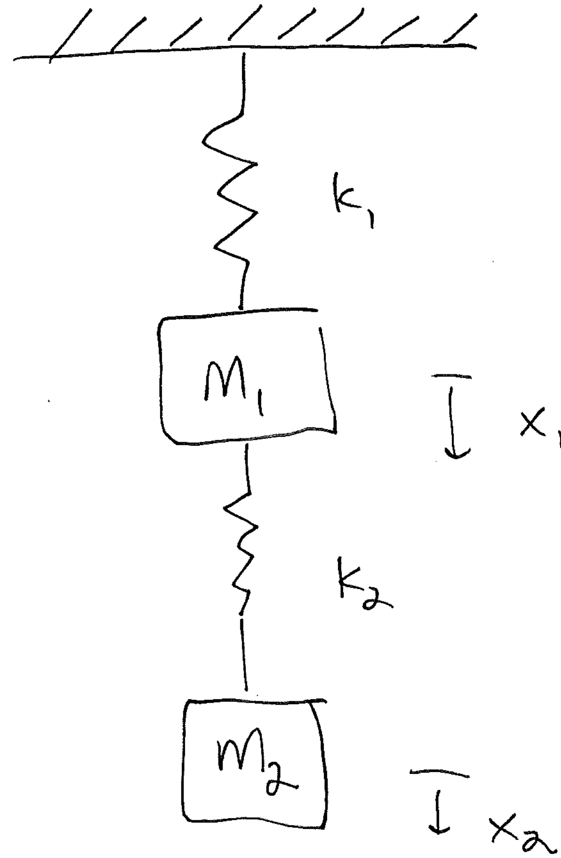
Ex 3: $\ddot{x} + \dot{x} - 2x = 0 \qquad x(0) = 3 \; ; \; \dot{x}(0) = 0$

$$\lambda^2 + \lambda - 2 = 0 \Longrightarrow (\lambda + 2)(\lambda - 1) = 0$$

$$\Longrightarrow \lambda = 1, -2$$

$$\boxed{x(t) = 2e^{t} + e^{-2t}}$$ Unstable: e^{t} dominates

```
clear all, close all, clc

t = 0:.01:10;

y0 = [2; -3];


x = exp(-t)+exp(-2*t);
plot(t,x,'k');
hold on

A = [0 1;
    -2 -3];
[t,y] = ode45(@(t,y) A*y, t, y0);
plot(t,y(:,1),'r--')
xlabel('Time [s]')
ylabel('Solution x')
legend('Analytic','ODE45')

%% eigenvalues of A should be roots of characteristic equation!
eig(A)
```



$$\begin{cases} m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0 & (1) \\ m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0 & (2) \end{cases}$$

$\Longrightarrow$ Single 4^{th} order equation

or 4 first order equations

Solve (1) for $x_2 = f(x_1)$

Take 2 derivatives $\ddot{x}_2 = \frac{d^2}{dt^2} f(x_1)$

and plug in to (2) !

Higher Order Systems:

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$$

- OR -

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

Try $x(t) = e^{\lambda t}$ $\left(\text{note } \frac{d^n x}{dt^n} = \lambda^n e^{\lambda t} = \lambda^n x(t)\right)$

$$\left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda + a_0\right)\underbrace{x(t)}_{e^{\lambda t}} = 0$$

$$\Longrightarrow \boxed{a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0}$$

Characteristic Equation!

In general, n solutions: $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_n e^{\lambda_n t}}$$

Need n initial conditions to determine constants $\{c_k\}_{k=1}^n$

Usually, use $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix}$ to solve for $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$

Note: you can always divide everything by a_n in top equation...

High Order ODE $\Longrightarrow$ System of 1st Order ODEs

$$x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

Introduce new variables

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \\ x_4 &= \dddot{x} \\ &\vdots \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -\left[a_{n-1} x_n + a_{n-2} x_{n-1} + \dots + a_2 x_3 + a_1 x_2 + a_0 x_1\right] \end{aligned}$$

negative sign!

As a matrix system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\Longrightarrow \boxed{\dot{\underline{x}} = \underline{\underline{A}}\, \underline{x}}$$

Claim: $\det(A - \lambda I) = 0$

is equal to characteristic equation.

Example: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -\lambda - a_2 \end{bmatrix}$$

$$= -\lambda\left[\lambda^2 + \lambda a_2 + a_1\right] - 1 \cdot \left[a_0\right] + 0$$

$$= \underbrace{-\lambda^3 - \lambda^2 a_2 - \lambda a_1 - a_0 = 0}$$

$$\Rightarrow \boxed{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0}$$

Characteristic Equation

We want to solve general systems of equations:

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}.$$

Case 1: Uncoupled dynamics

Different animals in a zoo (separated populations)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{x}_1 = \lambda_1 x_1 \Rightarrow x_1 = e^{\lambda_1 t} x_1(0)$$
$$\dot{x}_2 = \lambda_2 x_2 \Rightarrow x_2 = e^{\lambda_2 t} x_2(0)$$
$$\vdots$$
$$\dot{x}_n = \lambda_n x_n \Rightarrow x_n = e^{\lambda_n t} x_n(0)$$

$$\Downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}(t) = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}}_{e^{\underline{\underline{A}}t}}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}(t=0)$$

(matrix exponential easy for diagonal A...