L04 : Oct. 1, 2014 ME564, Fall 2014

Overview of Topics

1. Harmonic oscillator : $\ddot{x} + x = 0$

   (a) Taylor series

   (b) Try $x(t) = e^{\lambda t}$

   (c) Suspend Variables

   $$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -x\end{aligned} \qquad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \qquad \text{(Linear!)}$$

2. Damped oscillator : $m\ddot{x} + \delta\dot{x} + kx = 0$

   (a) Try $x(t) = e^{\lambda t}$ (again!)

   : Characteristic polynomial

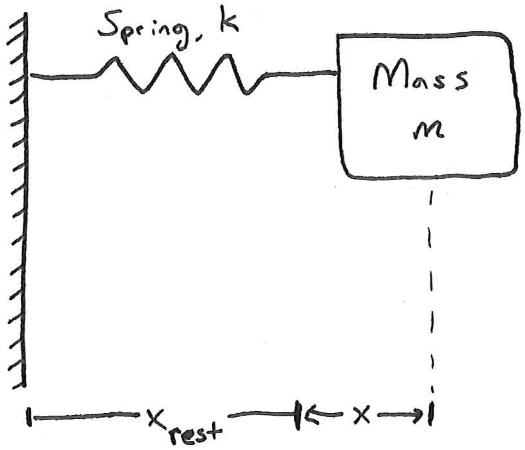
   $$m\lambda^2 + \delta\lambda + k = 0$$

   (b) Plot in Matlab

3. Higher order systems : $a_n x^{(n)} + \dots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$

4. Matrix systems of ODEs

Second-order systems



Newton's 2nd Law:

$$F = ma$$
$$= m\ddot{x}$$

$$\Longrightarrow \boxed{m\ddot{x} = -kx}$$

x is the displacement of the mass from a rest position x_{rest}, where spring exerts no net force.

First consider $k = m = 1 \Longrightarrow \boxed{\ddot{x} = -x}$ end L03

L04 start.

Method 1: Guess!

$$x(t) = \cos(t)\, x(0)$$
$$\dot{x}(t) = -\sin(t)\, x(0)$$
$$\ddot{x}(t) = -\cos(t)\, x(0) \quad \checkmark \; \ddot{x} = -x$$

For general m, k:

$$x(t) = \cos\left(\sqrt{k/m}\; t\right) x(0)$$

so frequency of oscillation is $\omega = \sqrt{k/m}$

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$$\underline{\underline{\ddot{x} = -x}}$$

Method 2 : Taylor Series

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \dots$$

$$\dot{x}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \dots$$

$$\ddot{x}(t) = 2c_2 + 3\cdot 2 c_3 t + 4\cdot 3 c_4 t^2 + 5\cdot 4 c_5 t^3 + \dots$$

$$\ddot{x} = -x \implies 2c_2 = -c_0 = -x(0) \implies c_2 = -\frac{1}{2} x(0)$$

$$3\cdot 2 c_3 = -c_1 = \dot{x}(0) \implies c_3 = \frac{-1}{3!} \dot{x}(0)$$

$$4\cdot 3 c_4 = -c_2 \implies c_4 = \frac{1}{4!} x(0)$$

$$c_5 = \frac{1}{5!} \dot{x}(0)$$

$\vdots$

$c_0 = x(0) \leftarrow$ initial position

$c_1 = \dot{x}(0) \leftarrow$ initial velocity

Say $\dot{x}(0) = 0$, so all odd coefficients $= 0$.

$$x(t) = x(0) - \frac{t^2}{2!} x(0) + \frac{t^4}{4!} x(0) - \dots$$

$$\implies \boxed{x(t) = \cos(t)\, x(0)}$$

$$\ddot{x} = -x$$

Method 2: Guess Again!

What function, when taking multiple derivatives, is similar to itself, up to a constant?

Answer: $x(t) = e^{\lambda t}$

$$\dot{x} = \lambda e^{\lambda t}$$
$$\ddot{x} = \lambda^2 e^{\lambda t} \xrightarrow{\ddot{x} = -x} \lambda^2 e^{\lambda t} = -e^{\lambda t} \implies \underline{\underline{\lambda^2 = -1}}$$

$\lambda^2 = -1$ for $\underline{\underline{\lambda = \pm i}}$

(*)
$$\left[\begin{aligned} x(t) &= c_1 e^{it} + c_2 e^{-it} \\ &= (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t) \end{aligned}\right.$$

Initial Condition(s).

$x(0) = c_1 + c_2$

Need another IC.

$\dot{x}(0) = i(c_1 - c_2) = 0$

$c_1 = c_2$

$\implies \underline{c_1 = c_2 = x(0)/2}$

$$\boxed{x(t) = x(0)\cos(t)}$$

$\ddot{x} = -x$

Method 4 : Suspend variables & solve as linear system

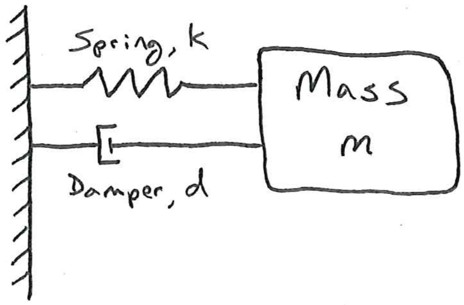
$$\dot{x} = v \xleftarrow{} \text{new variable}$$

$$\dot{v} = -x$$

$$\Longrightarrow \quad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

Much more on this later!

Example Damped Harmonic Oscillator



$$F = ma$$

$$m\ddot{x} = -kx - d\dot{x}$$

$$\Longrightarrow \quad m\ddot{x} + d\dot{x} + kx = 0$$

Try $\quad x(t) = e^{\lambda t} \quad \ldots$

$$\dot{x}(t) = \lambda e^{\lambda t}$$

$$\ddot{x}(t) = \lambda^2 e^{\lambda t}$$

$$\Longrightarrow \quad m\lambda^2 e^{\lambda t} + d\lambda e^{\lambda t} + k e^{\lambda t} = 0$$

$$\Longrightarrow \quad \left[m\lambda^2 + d\lambda + k\right] e^{\lambda t} = 0$$

$$\Longrightarrow \quad m\lambda^2 + d\lambda + k = 0$$

Let $\quad d/m = \zeta \quad$ and $\quad k/m = \omega^2$, so

$$\Longrightarrow \quad \lambda^2 + \zeta\lambda + \omega^2 = 0$$

$$\Longrightarrow \quad \boxed{\lambda = \frac{-\zeta \pm \sqrt{\zeta^2 - 4\omega^2}}{2}}$$

2 solutions (+/−)

λ_1 and λ_2.

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Need to initial conditions for c_1, c_2.

Higher Order Systems:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$$

- OR -

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

Try $x(t) = e^{\lambda t}$ (note $\frac{d^n x}{dt^n} = \lambda^n e^{\lambda t}$
$= \lambda^n x(t)$)

$$\left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0\right) \underbrace{x(t)}_{e^{\lambda t}} = 0$$

$$\Longrightarrow \boxed{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0}$$

Characteristic Equation!

In general, n solutions: $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_n e^{\lambda_n t}}$$

Need n initial conditions to determine constants $\{c_k\}_{k=1}^{n}$

Usually, use $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix}$ to solve for $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$

Note: you can always divide everything by a_n in top equation...

High Order ODE $\Longrightarrow$ System of 1st Order ODEs

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

Introduce new variables

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \\ x_4 &= \dddot{x} \\ &\vdots \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -\left[a_{n-1}x_n + a_{n-2}x_{n-1} + \dots + a_2 x_3 + a_1 x_2 + a_0 x_1\right] \end{aligned}$$

negative sign!

As a matrix system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\Longrightarrow \boxed{\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}}$$