## Overview of Topics

1. Taylor Series and $\dot{x} = \lambda x$
2. What <u>is</u> a Taylor Series (Matlab)
3. Second order systems

$$\ddot{x} + x = 0$$

- Harmonic oscillator, spring-mass
- Try Taylor series
- Suspend variables (Next Lecture?)

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x \end{aligned} \implies \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \quad \text{(Linear!)}$$

Next Lecture

$$\dot{x} = ax \quad \Rightarrow \quad x(t) = e^{at}x(0)$$

What is $e^t$?

Taylor Series Solution to $\dot{x} = ax$

$$x = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \dots$$

$$\dot{x} = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^4 + \dots$$

$$ax = ac_0 + ac_1 t + ac_2 t^2 + ac_3 t^3 + ac_4 t^4 + \dots$$

$\dot{x} = ax$ match each power of "$t$"

$\underline{t^0 = 1}$ : $c_1 = ac_0 = ax_0$ since $x(0) = c_0$

$= ax_0$

$\underline{t^1}$ : $2c_2 = ac_1 \Rightarrow c_2 = \frac{1}{2}a^2 x_0$

$\underline{t^2}$ : $3c_3 = ac_2 \Rightarrow c_3 = \frac{1}{3 \cdot 2}a^3 x_0$

$\underline{t^3}$ : $4c_4 = ac_3 \Rightarrow c_4 = \frac{1}{4 \cdot 3 \cdot 2}a^4 x_0$

$\vdots$

$$c_k = \frac{1}{k!}a^k x_0$$

$$x(t) = x_0 + at\,x_0 + \frac{a^2t^2}{2!}x_0 + \frac{a^3t^3}{3!}x_0 + \dots + \frac{a^kt^k}{k!}x_0 + \dots$$

$$= e^{at}x_0.$$

# Taylor Series

A function $f(x+\Delta x)$ may be Taylor expanded about a base point $x$ (assuming $f$ is smooth at $x$)

$$f(x+\Delta x) = f(x) + \frac{df}{dx}(x)\cdot\Delta x + \frac{1}{2!}\frac{d^2 f}{dx^2}(x)\cdot\Delta x^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}(x)\cdot\Delta x^3 + \underbrace{h.o.t.}_{\mathcal{O}(\Delta x^4)}$$

h.o.t = higher order terms

OR $f(x)$ expanded about a point "a":

$$f(x) = f(a) + \frac{df}{dx}(a)(x-a) + \frac{1}{2!}\frac{d^2 f}{dx^2}(a)(x-a)^2 + \ldots \text{ h.o.t.}$$

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Example: $f(x) = \sin(x)$

Taylor expand about $x=0$ (Maclaurin series)

$$f(x) = \sin(0) + x\cos(0) - \frac{x^2}{2!}\sin(0) - \frac{x^3}{3!}\cos(0) + \frac{x^4}{4!}\sin(0) + \frac{x^5}{5!}\cos(0) + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ forever.}$$

(MATLAB EXAMPLE)

Example: $f(x) = \cos(x)$

$$f(x) = \cos(0) - x\sin(0) - \frac{x^2}{2!}\cos(0) + \frac{x^3}{3!}\sin(0) + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

```
clear all; close all
x = -10:.01:10;
y = sin(x);
plot(x,y,'k','LineWidth',2)
axis([-10 10 -10 10])
grid on, hold on

%% First-order Taylor expansion
P = [1 0]; % x + 0;
yT1 = polyval(P,x);
plot(x,yT1,'b--','LineWidth',1.2)

%% Third-order Taylor expansion
P = [-1/factorial(3) 0 1 0]; % -(1/3!)x^3 + x + 0;
yT3 = polyval(P,x);
plot(x,yT3,'r--','LineWidth',1.2)

%% Fifth-order Taylor expansion
P = [1/factorial(5) 0 -1/factorial(3) 0 1 0]; % -(1/3!)x^3 + x + 0;
yT5 = polyval(P,x);
plot(x,yT5,'g--','LineWidth',1.2)

%% Seventh-order Taylor expansion
P = [-1/factorial(7) 0 1/factorial(5) 0 -1/factorial(3) 0 1 0]; % -(1/3!)x^3 + x + 0;
yT7 = polyval(P,x);
plot(x,yT7,'m--','LineWidth',1.2)

%% Ninth-order Taylor expansion
P = [1/factorial(9) 0 -1/factorial(7) 0 1/factorial(5) 0 -1/factorial(3) 0 1 0]; % -(1/3!)↙
x^3 + x + 0;
yT9 = polyval(P,x);
plot(x,yT9,'c--','LineWidth',1.2)
```

# Taylor Series Example: $f(x) = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \text{ h.o.t.} \quad = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$$

Even terms $= \cos(x)$

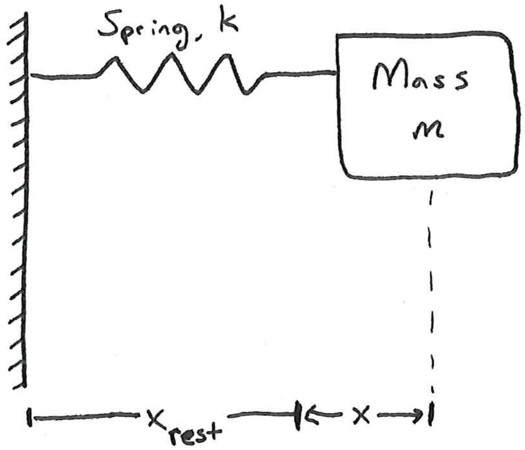
Odd terms $= i\sin(x)$

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

Euler's Formula!

FYI ... anything with Euler's name is a <u>big</u> <u>deal</u>.

# Second-order systems




Newton's 2nd Law:

$F = ma$

$= m\ddot{x}$

$\Longrightarrow \boxed{m\ddot{x} = -kx}$

$x$ is the displacement of the mass from a rest position $x_{rest}$, where spring exerts no net force.

First consider $k = m = 1 \Longrightarrow \boxed{\ddot{x} = -x}$ end L03

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L04 start.

Method 1: Guess!

$x(t) = \cos(t)\, x(0)$

$\dot{x}(t) = -\sin(t)\, x(0)$

$\ddot{x}(t) = -\cos(t)\, x(0)$ ✓ $\ddot{x} = -x$

For general $m, k$:

$$x(t) = \cos\left(\sqrt{k/m}\; t\right) x(0)$$

so frequency of oscillation is $\omega = \sqrt{k/m}$