ME564 - Autumn 2022 **Due: 5:00:00pm Friday November 4^{th}, 2022**
Homework 3

Exercise 3–1: Please solve the following differential equation (with initial conditions) for the three cases below (by hand!). You may use whatever method you find simplest. You may check your work on a computer.

$$\begin{aligned} \ddot{x} + 5\dot{x} + 6x &= f(t), \\ x(0) &= \frac{1}{2}, \\ \dot{x}(0) &= -1. \end{aligned}$$

(a) For $f(t) = 0$. Note that this is just the unforced ODE $\ddot{x} + 5\dot{x} + 6x = 0$.

(b) For $f(t) = e^{-t}$.

(c) For $f(t) = 50\cos(t)$. Hint: try a particular solution $x_P = A\cos(t) + B\sin(t)$ and solve for A and B.

In all cases, be sure to make sure that your initial conditions are still satisfied!

Exercise 3–2: Consider the following nonlinear, forced ODE:

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma\cos(\omega t)$$

For $\beta = -1$, $\alpha = 1$ and $\gamma = 0$, we may write this as a system of first order differential equations as

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\delta v + x - x^3 \end{aligned}$$

You may assume that $\delta \geq 0$.

(a) Write down all of the fixed points of the system of equations.

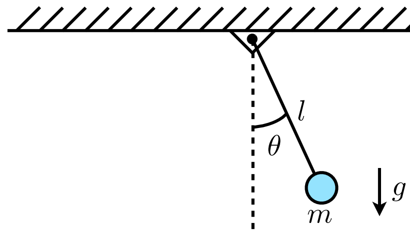
(b) For each fixed point, write down the linearized equations near the fixed point.

(c) For each linearized system, set $\delta = 0$ and determine what type of fixed point it is (source, sink, center, spiral, saddle, etc.) and what the stability is.

(d) Describe in words how these fixed points will change if δ is a small positive number.

(e) Please sketch the phase portrait (i.e. trajectories in the x-v plane) for the system with a small positive δ. Pick one of the stable fixed points and sketch the set of initial conditions that will eventually end up near this fixed point.

Exercise 3–3: Consider the schematic of the single pendulum.



The kinetic energy T and potential energy V may be written as:

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$
$$V = -gml\cos(\theta)$$

The Lagrangian $\mathcal{L}$ is given by $\mathcal{L} = T - V$, and the Euler-Lagrange equations for the motion of the pendulum are given by the following second order differential equation in θ:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

Write down the second order ODE using the specific T and V defined above. Please write this ODE in the form $\ddot{\theta} = f(\theta, \dot{\theta})$. Notice that this ODE is not linear!!

Now you may assume that $l = m = g = 1$ for the remainder of the problem.

You may still suspend variables to get a system of two first order (nonlinear) ODEs by writing the ODE as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = f(\theta, \omega)$$

(a) What are the fixed points of this system where all derivatives are zero?

(b) Write down the linearized equations in a neighborhood of each fixed point and determine the linear stability. You may formally linearize the nonlinear ODE or you may use a small angle approximation for $\sin(\theta)$; the two approaches are equivalent.

(c) Compute the eigenvalues and eigenvectors for each fixed point.

(d) Sketch the linearized phase portrait in a small neighborhood around each of these fixed points.

(e) How do the above answers match your physical intuition about the fixed points of this system?

Exercise 3–4: Consider the following physical system of a bead constrained to move in the potential field $\mathbb{V}(x)$. Note: there are no equations given, and so you must use physical intuition.

- Please sketch the phase portrait (position x vs. velocity v) for the Please try to make your sketch as accurate as possible using as much information about the potential as you can. You may draw the phase portrait without any damping (i.e., no friction).
- Comment briefly on how the plot will change if we turn on friction. (you do not need to sketch anything)
- How many fixed points does the system have and what is their stability?
- What can we say about the eigenvalues of the linearization around each point?