**ME564 - Autumn 2022** **Due: 5:00:00pm Friday October $21^{\text{st}}$, 2022**
**Homework 2**

---

**Exercise 2–1:** Please solve the following differential equation (by hand) and describe the long-time behavior of the system:

(a) $\ddot{x} + 4\dot{x} + 3x = 0$, with initial conditions $x(0) = 0$ and $\dot{x}(0) = 4$.

(b) $\ddot{x} - 4x = 0$, with initial conditions $x(0) = 4$ and $\dot{x}(0) = -4$.

---

**Exercise 2–2:** Write down the second-order differential equation $\ddot{x} + K_1\dot{x} + K_2 x = 0$ that has eigenvalues $\lambda = C$ and $\lambda = D$. If the solution is $x(t) = Ae^{Ct} + Be^{Dt}$, what are the initial conditions?

---

**Exercise 2–3:** Consider the following ODE:

$$\ddot{x} + 7\dot{x} + 10x = 0.$$

(a) Solve this ODE by substituting in $x(t) = e^{\lambda t}$ and finding roots of the characteristic polynomial.

(b) Now, write this second order ODE as a system of first order equations, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, by introducing a new variable $v = \dot{x}$. Solve the linear system by decomposing the matrix $\mathbf{A}$ into the eigenvalues and eigenvectors. Hint: computing the inverse of the matrix of eigenvectors is easier if they are not normalized (i.e., both of my eigenvectors have integer entries).

(c) For both of the solutions in parts (a) and (b), write down the specific solution for an initial condition $x(0) = 1$ and $\dot{x}(0) = 1$.

(d) What is the long-time behavior of this system?

---

**Exercise 2–4:** Write the following ODE as a system of first order ODEs:

$$\dddot{x} + 2\ddot{x} - \dot{x} - 2x = 0$$

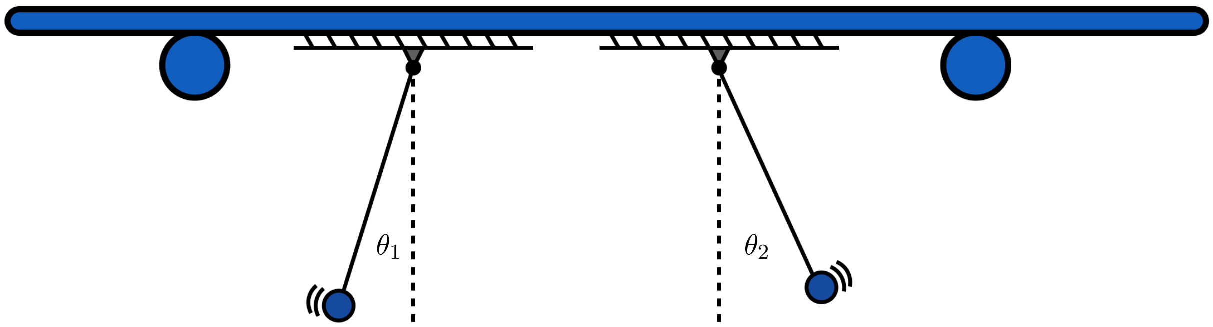
What are the eigenvalues of the system of ODEs? (it is OK to use a computer).

What are the long-time behaviors of this system for the following two different initial conditions: $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$?

Please plot the response for `t=0:0.01:10` using a Runge-Kutta 4th order integrator for each of these initial conditions. Do the plots agree with your calculations? Please briefly explain your observations in 1-2 sentences.

---

**Exercise 2–5:** Consider the system of weakly coupled pendulua (equations below). Both pendula are mounted to a board that is placed on rollers, so that it can move from side to side, slightly.




$$\ddot{\theta}_1 = -\omega_1^2\theta_1 + \epsilon(\theta_2 - \theta_1)$$
$$\ddot{\theta}_2 = -\omega_2^2\theta_2 + \epsilon(\theta_1 - \theta_2).$$

Write these coupled second order linear differential equations as

(a) A single fourth order ODE in $\theta_1$

(b) A single fourth order ODE in $\theta_2$

(c) A system of four coupled linear ODEs in terms of the angular positions and velocities of each pendulum. Please write this as a matrix ODE.

(d) Now, assume that $\omega_1 = 1$ and $\omega_2 = 1.5$. Increase $\epsilon$ from 0 to 0.5 (in increments of 0.005), and compute the eigenvalues of the system of equations. Plot the two frequencies as a function of $\epsilon$. Now plot the difference of the two frequencies agains $\epsilon$. Explain what you see.

(e) At what value of $\epsilon$ will the frequencies of the coupled system be equal for $\omega_1 = 1$ and $\omega_2 = 1.5$?

(f) At what value of $\epsilon$ will the frequencies of the coupled system be equal for generic $\omega_1$ and $\omega_2$?

**Exercise 2–6:** Consider the following nonlinear ODE:

$$\frac{dx}{dt} = x^2$$

(a) Find the solution $x(t)$ using any method you prefer.

(b) A major problem in engineering mathematics is to find a good coordinate system where complex system become *simple*. There is a change of coordinates $y(x)$ that makes this system linear, with dynamics

$$\frac{dy}{dt} = y.$$

Using the chain rule, we find that:

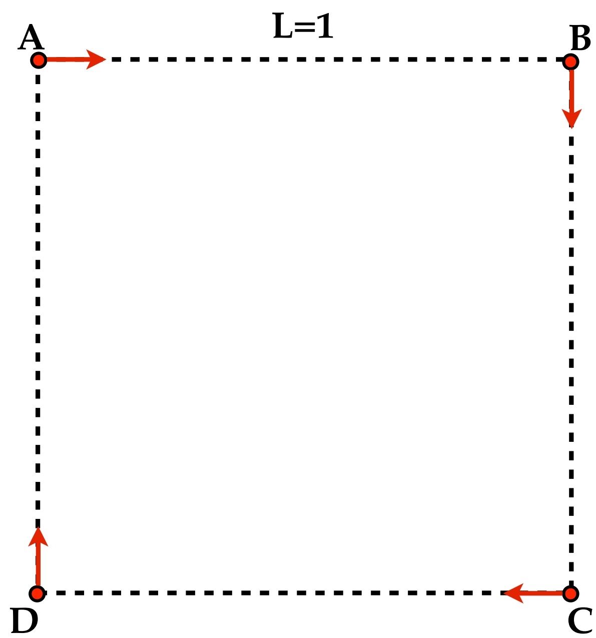
$$\frac{dy(x(t))}{dt} = \frac{\partial y}{\partial x}\dot{x} = y(x).$$

Plugging in $\dot{x} = x^2$, we get

$$\frac{\partial y}{\partial x}x^2 = y(x).$$

Solve for $y(x)$ using a series expansion. Show that a simple Taylor series will not work. Then try a series that includes negative powers of $x$: $y(x) = \cdots + c_{-2}x^{-2} + c_{-1}x^{-1} + c_0 + x_1 x + x_2 x^2 + \cdots$.

**(Optional, for Fun)** There are four boats at four corners of a square. Each boat starts motoring towards the neighboring boat in the clockwise direction (so $A$ pursues $B$, $B$ pursues $C$, $C$ pursues $D$ and $D$ pursues $A$). As the boats begin to move, they always move in a direction pointing towards the boat they are pursuing, so that they eventually meet in the middle. Each boat motors at a constant speed of 1 mile per hour in pursuit of its neighbor. The square has sides of length 1 mile.




- Write down a differential equation that describes the motion of a boat.
- Sketch the motion of the boats in time.
- How long until the boats meet in the middle?

Hint: Try to find any symmetry or convenient coordinates to use that make the problem simpler. The simplest solution may surprise you! You might also want to try and draw the situation and explain your reasoning.