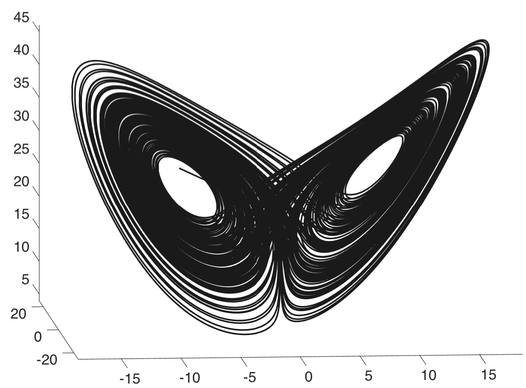
Exercise 4–1: Consider the chaotic Lorenz system:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z.\end{aligned}$$

A chaotic trajectory for parameters $\sigma = 10, \rho = 28$, and $\beta = 8/3$ are shown below.



This system has three fixed points at: $(0, 0, 0)$, $(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$, and $(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)$.

For each fixed point, linearize the Lorenz equations for a neighborhood around the fixed point and determine the eigenvalues and stability (You can use MATLAB for this part). Draw a rough sketch of the phase portrait around these fixed points.

Exercise 4–2: Consider the same chaotic Lorenz system from the previous exercise.

Now, determine the eigenvalues and stability for each fixed point for $\rho \in \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$. (no need to sketch phase portrait for all cases)

For each value of ρ, simulate and plot the trajectory of the ODE using `ode45` in MATLAB with initial condition $\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$ for `t=0:.01:20`.

You will want to create a MATLAB .m file called `lorenz.m` with the following inputs and outputs:

```
function dy = lorenz(t,y,sigma,rho,beta)
```

Do the trajectories for various ρ agree with the linearization results?

Exercise 4–3: Derive a $\mathcal{O}(\Delta t^2)$ accurate backward difference derivative for the function $f(t)$ at t using the following three *measurements* or data points:

$$f(t), f(t-\Delta t), \text{ and } f(t-2\Delta t)$$

Please show that this scheme is really $\mathcal{O}(\Delta t^2)$ accurate using the Taylor series expansions.

Exercise 4–4: Consider the differential equation for the Van der Pol oscillator (use `ode45`)

$$\ddot{y} + \epsilon(y^2 - 1)\dot{y} + y = 0$$

which has a nonlinear damping term $\epsilon(y^2 - 1)\dot{y}$.

(a) Write this ODE as a system of first order differential equations.

(b) Analyze the stability of the fixed point at $y = \dot{y} = 0$ for $\epsilon > 0$.

(c) For $\epsilon = 0.1$, solve the equation for `t=0:0.1:30` for initial conditions $y(0) = 0.1$ and $\dot{y}(0) = -1$. Repeat for $\epsilon = 1$ and $\epsilon = 20$. Plot the phase portrait (y vs. $\dot{y}$) for each of the three cases to see the various behaviors.

You will want to create a MATLAB .m file called `vanderpol.m` with the following inputs and outputs:

```
function dy = vanderpol(t,y,eps)
```
